ELEMENTS OF

ELECTROMAGNETIC THEORY

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BY

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MY REVERED FRIEND

PROFESSOR FRANCIS H. SMITH, LL.D.

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PREFACE.

In this treatise I have tried to present in systematic and definite form a simple, rigorous, and thoroughly modern introduction to the fundamental principles of electromagnetic theory, together with some of the simpler of their more interesting and important non-technical applications. The work makes no pretense to completeness, but is written for the serious student of physics, who will make liberal use of more detailed treatises, of hand-books, and of journals, as occasion demands.

I am of course indebted to many books and memoirs. My obligations are especially great, as the most cursory examination of the book will show, to the works of Maxwell, Heaviside, and Poynting. I am also much indebted to Professor A. G. Webster for the use of a number of excellent diagrams from his treatise on electrical theory.

S. J. BARNETT.

STANFORD UNIVERSITY, CALIFORNIA
June, 1903.

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CHAPTER I.

GENERAL ELECTROSTATIC THEORY.

1. Electrification by Contact. Positive and Negative Charges. Let one end of an ebonite rod and a dry woolen cloth be rubbed or strongly pressed together and then separated; and let a second rod and cloth be treated in the same way: The rubbed part of each cloth will be found, on trial, to be attracted toward the rubbed part of each rod, while the rubbed part of each cloth will be repelled from the rubbed part of the other cloth, and the rubbed part of each rod from the rubbed part of the other rod.

These are examples of *electric* phenomena. The region in which they are manifested is called an *electric field* (§ 11), and the medium which permeates this region — air and æther in the above case — and through which electric influences are transmitted is called a *dielectric*. The parts of the ebonite and wool rubbed together are said to be *electrified*, or to possess *electric charges*. The two pieces of woolen cloth are said to have *like* charges, since they were similarly treated and since what is repelled from one is repelled from the other, and what is attracted toward one is attracted toward the other. Similarly, the two ebonite rods are said to have like charges. But the wool and the ebonite are said to have *unlike* or *opposite* charges, since what is repelled from one is attracted toward the other.

Like ebonite and wool, any two different substances, or portions of the same substance in different physical conditions, exhibit electric properties after intimate contact and separation. One of the bodies behaves like ebonite rubbed with wool, the other like the wool.

An electric charge like that of wool after contact with ebonite is called a *positive* charge, and a charge like that of the ebonite, a *negative* charge. The terms positive and negative are justified by the opposite properties of the two kinds of electrification, but there is no reason except convention and resulting convenience why the two terms should not be interchanged.

In addition to the forces between electrified bodies, forces are found to exist, in general, between an electrified body and an insulator (§ 2) not electrified (Chapters IV. and VI.).

2. Conductors and Insulators. Electrification by Conduction. A rod of ebonite electrified at one end exhibits electric properties only at that end; while a rod of metal, held by an ebonite handle and electrified at one end, becomes electrified at once (apparently) all over its surface. Substances like the metals, by which an electric charge is distributed with extreme rapidity, so as to come into a state of equilibrium within (usually) a small fraction of a second, are called *electric conductors*. A body charged by connection with an electrified body through a conductor, like the far end of the metal rod mentioned above, is said to be *electrified by conduction*. Substances like ebonite, over or through which an electric charge is transferred only with extreme slowness, are called electric *insulators* or *non-conductors*.

Among ordinary molecular substances *perfect* insulators and *perfect* conductors do not exist, no such substance completely and for an indefinite time preventing all transfer of electrification, and all offering more or less obstruction to such transfer. There is every reason to believe, however, that free æther (a "vacuum") and clean dry gases containing no (electrolytically) dissociated molecules have the properties of a perfect insulator (Chapter IX.).

Among substances possessing high conductivity are the metals, graphite, and salt or acid solutions; among those with high insulating properties are (undissociated) gases, fused quartz (cold and in the solid state), ebonite, cold glass, silk, and wool. A substance which is an excellent insulator in one condition, however, may in another condition have the properties of a conductor. Thus cold glass is an excellent insulator, but as the temperature is raised its insulating properties disappear. Also, under very great electric stress the insulating properties of all molecular substances break down.

A body completely surrounded with insulators is said to be *insulated.*

A conductor can be completely *discharged* by bringing it into contact at any *one* point with the inner surface of a hollow closed conductor (§ 4), such as the walls of the room within which the experiments are performed, provided there are no (insulated) electrified bodies within. When connected to the walls of the room, or the earth, the conductor is said to be *earthed.* From an insulator the electrification can be entirely removed only by applying a conductor at every electrified point, *e. g.*, by immersing it in a conducting gas or liquid.

3. Electrification by Induction. An insulated conductor, when brought near an electrified body, *i. e.*, into an electric field, itself becomes electrified. Examined by the methods of § 1, the charges of the more remote and nearer ends of the conductor are found to be similar and opposite, respectively, to that of the original electrified body. A conductor electrified in this manner is said to be electrified by *induction.*

If the conductor, while still insulated, is removed from the electric field, all signs of electrification disappear. But if, while still in the field, it is connected with the walls, or *earthed*, the electrification similar to that of the original charged body disappears, while the opposite electrification of the near end remains. If the conductor is now insulated and removed from the original

electric field, this charge becomes more evenly distributed over its surface (§ 42). In this manner any number of conductors may be given charges opposite to that of a given electrified body without, as may be proved by the method of § 5, diminishing or increasing the latter's electrification.

4–8. Experiments with Hollow Closed Conductors. Electric Screens. Let A denote an insulated hollow conductor having a closely fitting conducting lid, B, with an insulating handle. Let A be connected with an electroscope or electrometer (Chapter III.), C, by means of which any change in the state of electrification of its exterior (or interior) surface may be detected; and let A be kept closed except when another body is being introduced into its cavity, or removed therefrom, or its position inside (or outside) altered.

4. (1) Let the electrometer be placed outside of A. If A is initially unelectrified, and an insulated unelectrified conductor, D, is now introduced into A without touching it, the inner and outer surfaces of A will become electrified by induction (§ 3) with charges opposite and similar, respectively, to that of D. And the electrification of the external surface, as indicated by the electrometer, will be found to remain absolutely unaltered howsoever D is moved about within, even when it is brought into contact with A; but D, on being insulated after contact, and then removed from A's interior, will be found completely discharged. This process may be repeated indefinitely, D always becoming completely discharged on coming into contact with the inner surface of A. If A is initially electrified in any manner, the phenomena will be precisely the same, except that the external electrification and the corresponding indication of the electrometer will be different.

(2) Let the electrometer be placed within A, either connected with A metallically, or insulated therefrom. In this case it will be found that if there are insulated charged bodies within A, the electrometer will give a certain deflection; that if there are no

insulated electrified bodies within A, the electrometer will give no deflection; and that its indication in either case will remain absolutely unaltered howsoever the electrification of the exterior of A or of external bodies is changed, even if A is connected to the walls of the room.

These experiments are due to Faraday, who constructed for the purpose of performing (2) a closed conductor large enough to enable him to make the observations while himself inside the cavity.

An experiment similar in principle to those of Faraday, but less general, performed earlier by Cavendish and repeated later by Maxwell with all the precision of modern investigation, gave identical results.

From the experiments just described it follows that, when there is electrical equilibrium,

1. *An electric charge cannot exist in the substance of a conductor, or on the inner surface of a hollow closed conductor* (unless there are insulated electrified bodies within). For D, on being removed from A, of whose substance it formed a part, electrically, while in contact, was always unelectrified.

2. *An electric field* (§ 11) *does not exist within the hollow of a closed conductor* (unless there are charges inside). For in (2) the electrometer was unaffected (by induction or otherwise) no matter what the external electrification, except when there were insulated charges within.

3. *The electric charges and electric fields within and without a hollow closed conductor are absolutely independent of one another.* The conducting shell thus completely *screens* each of these regions from all static effects in the other.

4. *An electric field does not exist within the substance of a conductor.* See § 15.

5. Equal Charges. Two electric charges of the same sign are, by definition, of the same magnitude if they produce the same effect on the electrification of the vessel A when introduced in succession separately.

Similarly, two charges of opposite signs are, by definition, equal in magnitude if they produce no effect on the electrification of A when introduced simultaneously.

These definitions are independent of the particular closed conductor A used, as two charges defined as equal by means of one such vessel are found to remain equal when tested in the same way with any other hollow closed conductor.

6. Positive and Negative Charges are Always Developed Simultaneously in Equal Amounts. If two bodies electrified by contact are introduced into the vessel A simultaneously, the indication of the electrometer remains unaltered.

If an electrified body is insulated within A, and if an insulated uncharged conductor is then introduced in addition, the latter becomes electrified by induction, in conformity with § 3, but the indication of the electrometer remains unaltered.

In these cases, therefore, positive and negative charges are developed in equal amounts (§ 5); and in the same way it may be shown that this is always the case, howsoever the electrification is produced.

7. The Total Quantity of Electrification is Unaltered by Conduction. If the two insulated bodies of the last experiment are brought into contact with one another while inside the vessel A, or if they are brought into contact with the inner surface of A itself, conduction occurs, but no effect on the external electrification is produced. From this it follows that when conduction occurs, the total (algebraic) amount of electrification is unaltered.

Corollary. The charges induced on the inner and outer surfaces of A when an electrified body is introduced and insulated within, as in § 4, *are each of the same magnitude as that of the insulated body.* For when D touches A, the charges of D and the inner surface of A completely disappear by conduction, since D on removal is unelectrified; thus their algebraic sum is zero. And the (opposite) charges on the inner and outer surfaces, being induced, must, by § 6, be equal in magnitude.

8. Electric Charges of Both Kinds Measured in Terms of a Single Arbitrary Unit. In addition to the hollow conductor *A* of §§ 4–7, let there be provided another similar insulated vessel *B*, sufficiently large to admit *A* through its opening; and let the conductor *D* be given a certain charge (suppose positive for the sake of definiteness), which will be adopted as a provisional unit.

If now *D* is brought within *A* and kept insulated, the outer surface of *A* will have unit positive charge. If *A* is brought inside *B* and then into contact with it, this charge will disappear, as will also the charge induced on *B*'s inner surface, leaving the outside of *B* with unit positive charge. If *A* is now removed from *B*'s interior and then *D* from *A*, the negative charge induced on *A*'s inner surface will pass to the outer surface and will disappear when *A* is discharged. This complete process may be repeated any number of times. Each time *B* will acquire an additional unit positive charge, and thus may be given a measured positive charge which is any integral multiple of the original unit.

To give *B* a negative charge measured in terms of the same unit, the outer surface of *A* must be brought into contact with the inner surface of a hollow closed conductor after the introduction of *D*, when the positive charge will disappear from the outside, leaving unit negative charge upon the inner surface. When *D* is removed, this charge will pass to the outer surface of *A*, and will be given up wholly to *B* when *A* is brought into contact with *B*'s interior. *B* will now have unit negative charge, and by removing *A* and repeating the process may be given any number of units negative charge desired.

To obtain any submultiple, $1/n$, of the original charge, it is only necessary to arrange symmetrically in contact the original conductor *D* and $n - 1$ precisely similar and equal conductors, all other bodies, except the surrounding dielectric, supposed homogeneous and isotropic, being so remote as to have no appreciable effect. Then, by the principle of symmetry, each conductor will take $1/n$ of the original charge.

9. The Law of Coulomb. Let two small spherical insulated conductors which can be given any charge desired, measured in terms of some provisional unit by the methods of §§5 and 8, be so connected with a dynamometer, such as a gravity balance, that the force F between them can be measured as their charges, q_1 and q_2, the distance L between their centers, and the surrounding dielectric are varied. Then it is found by experiment that,

(1) However the distance L and the charges q_1 and q_2 are varied, provided all the experiments are performed in the same dielectric, and provided that this dielectric is homogeneous and isotropic and extends to a great distance on all sides of the electrified bodies, F is in the straight line joining the centers of the conductors; is directly proportional to the product of their charges, being repulsive (considered positive) when the charges are like and attractive (considered negative) when the charges are unlike, as already known from §1; and the greater L in comparison with the linear dimensions of the charged bodies, the more nearly inversely proportional to L^2.

(2) In different dielectrics, with all other conditions the same, the force is different, and always less than in vacuo (free æther).

The general expression for F, when the linear dimensions of the (not necessarily spherical) charged bodies are negligible in comparison with their distance apart, is therefore

$$F = Aq_1q_2/cL^2 \qquad (1')$$

where c is a constant depending on the medium in which the experiments are performed, called its *permittivity* or *dielectric constant*, and A is a positive constant depending on the units in which q_1, q_2, L, F, and c are expressed.

(1′) expresses *Coulomb's law*.

The Rational Electrostatic Unit Charge. Unit Permittivity. In what follows, unless the contrary is stated, the centimeter will be used as unit length, the dyne as unit force, the permittivity of free æther, which will be denoted by c_0, as unit permittivity,

and as unit charge the charge which each of two indefinitely small bodies must have in order that when at a distance of 1 cm. apart in a vacuum the force between them may be $1/4\pi$ dyne. This unit charge is called by its originator, Oliver Heaviside, the *rational electrostatic unit charge*, and c_0 is called the *electrostatic unit permittivity*.

Methods of measuring permittivity are discussed in Chapter VII.

The conventions just made give, by the above equation, $A = 1/4\pi$, and the equation reduces to

$$F = q_1q_2/c4\pi L^2 \qquad (1)$$

which, in addition to being a particular case of (1′), is a particular case of (2).

The direct experimental investigation of the law of force is due to Coulomb, but is not capable of great precision. The law, as stated by Coulomb, is most satisfactorily established by the consideration that all experimental knowledge is in perfect accord with an electrical theory based largely upon the assumption that the laws expressed in (1) are exact.* A reason for the law of inverse squares and a justification of the term rational unit will be given in §§ 5, II., and 24.

The dimensions of electric charge and the other electric quantities, as well as other systems of units, will be considered in Chapter XIV.

For *rational electrostatic* the abbreviation *RES* will hereafter be employed.

10. If any one of the experiments described above is repeated in different dielectrics, the results in all cases will be identical, except that, in conformity with § 9, the force between two charged bodies will always depend on the surrounding dielectric.

* The common deduction of the law of inverse squares from the results of the Cavendish experiment cannot be accepted as valid. See *The Physical Review*, September, 1902, p. 175.

11. Electric Field. Electric Intensity. — Any region in which an electrified body is acted upon by a mechanical force in virtue of its charge, or in which an uncharged conductor is charged by induction, is called an *electric field*. Such a field exists, for example, around an electrified body (§ 1), but may also exist without the presence of electrification (Chapters VI. and XIII.).

As a result of experiment, it may be stated that the force F acting upon a small charged body, or small portion of a charged body, at any point of an electric field is proportional to its charge q — provided that the distribution of electric charge (real and apparent, Chapter IV.) originally accompanying the electric field remains undisturbed by the introduction of q. Expressed in the form of an equation, this relation is

$$F = Eq \tag{2}$$

where E is a constant for the given point of the field called the *electric intensity*, *electric force*, or *voltivity* at the point.

The conditions for the rigorous proof of this relation by direct experiment would be impossible to realise, and the remark at the close of §9 with reference to the establishment of Coulomb's law applies without alteration to (2).

As (2) shows, E is not a mere number, but a physical quantity specifying the state of the field and such that its product by an electric charge is a mechanical force. E is clearly a vector quantity, its direction being that of the force on a positively charged body, and its magnitude the number of dynes per unit charge. When q is expressed in the RES unit charge and F in dynes, E is said to be expressed in the *RES unit electric intensity*.

The term *electric field* is often used to denote the collective intensity in a region, instead of the region itself. The *direction of the field* at any point is the direction of the intensity, and the *strength of the field* is the magnitude of the intensity.

12. The Superposition of Electric Fields. — Experiment also shows that any number of electric fields (up to a certain limit,

when the dielectric breaks down and conduction occurs) may be superposed upon one another, the effect of each being independent of all the rest. Electric intensities, being vectors, may therefore be compounded like all other vectors for which the principle of superposition holds, the resultant intensity at any point being the geometric or vector sum of the component intensities.

An electric field is *uniform* if its intensity is the same at every point. Since E is a vector, this condition necessitates a constant direction as well as a constant magnitude. In most cases E varies from point to point. Examples of uniform and other electric fields, as well as of the superposition of electric fields, will be given below.

13. Electric Displacement or Induction. Electrisation. The physical nature of every electric quantity is at present unknown. Many phenomena, however, support the hypothesis that c is an elastic permittivity (*i. e.*, the reciprocal of an elastic modulus) and that E is an elastic stress. For the sake of constructing a mechanical conception of the electric field we shall provisionally *assume* c and E to be a permittivity and a stress, respectively. The so-called permittivity c will then be the actual permittivity of the æther or æther entangled in matter for the (unknown) kind of strain concerned.

Now, in the case of ordinary elastic substances subjected to slight mechanical strains we have, very approximately, the relation (Hooke's law): *strain/stress* = *1/modulus* = *permittivity*, or *strain* = *permittivity* × *stress*. If then c is a permittivity of a certain type and E a stress of the corresponding type, their product cE must measure the corresponding strain or displacement of the dielectric.

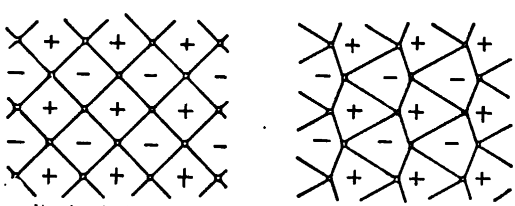
Whether this conception is correct or not, *the product* cE *is called* the *electric displacement* (also the *electric induction*), and is denoted by D. That is

$$D = cE \tag{3}$$

E being a vector, and c being the same for every direction of the intensity, since isotropic substances only are to be considered here, D is a vector with the same direction as that of E. When c and E are expressed in *RES* units, D is said to be expressed in the *RES unit displacement* (or *induction*).

A substance in which there is electric displacement is also said to be in a state of *electrisation*, or to be *electrised*. If the displacement and permittivity are uniform throughout, the electrisation is said to be *uniform*.

14. Mechanical Conception of the Electric Field. A definite conception of the electric field based on the *assumptions* made above will now be given. According to this conception (which leads to results by no means wholly consistent, however) the æther is the simplest possible kind of dielectric and is composed of two kinds of minute, incompressible, elastic cells, called



a. No electric displacement

b. Electric displacement directed to left

Fig. 1.

positive and negative cells, respectively, so arranged (in rows), Fig. 1, *a*, that only unlike kinds are in contact, and that no slip between adjacent cells is possible.

When the æther supports an electric field, the cells remain unchanged in volume, but their shapes are distorted and their centers of volume displaced, Fig. 1, *b*, the centers of the positive cells in the direction of the electric intensity, and the centers of the negative cells in the opposite direction. The electric displacement is measured by the relative linear displacement of the centers of volume of the cells of a positive row with reference to the centers of volume of the adjacent negative rows divided

by the distance between two adjacent rows. The electric intensity is the force per unit area in the direction of D acting upon the *positive cells*, or the force per unit area in the opposite direction to that of D acting upon the *negative cells*, in any plane passing through the direction of D. For small displacements, the displacement and intensity so measured will be proportional, as required by (3) in *all* cases. The total mechanical force acting upon the *whole* substance within any element of volume is zero.

From what precedes and from the nature of the distortion as shown in the figure, it is clear that there is a *tension* in the æther parallel to the intensity, and a *pressure* in all directions normal to the intensity. That this deduction from our mechanical conception is consistent with fact is demonstrated in §§ 40–41.

When the dielectric, instead of free æther, is a molecular substance permeated by æther, the same general conception is useful. Like the æther which permeates the matter, its molecules may be thought of as composed each of two constituents, positive and negative atoms, or atomic groups, or corpuscles (Chapter IX.), which suffer a displacement similar and in addition to that of the æther cells entangled among them. However this may be, the permittivity of all molecular substances yet investigated is greater than that of free æther. Thus, in ordinary matter a greater displacement than in free æther accompanies a given intensity.

In perfect insulators, according to our conception, the cells cannot slip over one another, and thus *elastic* displacement only can accompany electric intensity. In an imperfect insulator the cells can slip only with extreme slowness, and more slowly the more highly insulating the substance. In a conductor electric stress can exist only temporarily (unless an impressed electromotive force, Chapter VIII., is continuously acting), and is always accompanied by rapid slip. That the substance of a conductor cannot support electric displacement in a static field will be shown in § 15. The mechanical conception of electric conduction will receive further consideration later on (Chapter IX.).

15. Electric Displacement and Intensity Zero within a Conductor in a Static Field. We may now restate (4), § 4, as a corollary of (3), § 4: A static field cannot exist within the substance of a conductor. For the fields within and without a hollow closed conductor are absolutely independent of one another, however thin the conducting shell. Hence they cannot be *connected* by an electric field or electrically strained medium, and the whole substance of a conductor, except an extremely thin surface layer, is without electrical significance (in a static field). Thus the electric intensity and displacement in the outer region terminate at the outer surface of the conductor, and the electric intensity and displacement of the inner region (if the conductor is hollow and encloses insulated electrified bodies) terminate at the inner surface.

16. Lines and Tubes of Intensity, Displacement, etc. A line so drawn in an electric field as to have at every point along its length the direction of the electric intensity (electric force), electrisation, or displacement (induction) is called a *line of intensity* (*force*), *electrisation, or displacement* (*induction*).

A tubular surface the elements of which consist wholly of lines of intensity or induction (etc.) is called a *tube* of intensity or induction (etc.).

The strength of a tube of induction or displacement is defined in § 23.

17. Voltage, Electromotive Force, and Difference of Potential. The work done by the electric field in carrying an indefinitely small body with electric charge q along an element dL (Fig. 2) of a path L between two points P_1 and P_2 of an electric field, if dL makes an angle θ with the electric intensity E, is

$$dW = qE \cos \theta \, dL \tag{4}$$

and the total work done in carrying q along L from P_1 to P_2 is

$$W = q \int E \cos \theta dL \tag{5}$$

the integral being taken from P_1 to P_2. To carry q from P_2 to P_1 along the same path would of course require the expenditure of the same amount of work *against* the field by an outside agent.

In the same way the work done by the field in carrying the body with charge q from P_1 to P_2 along another path L' is

$$W' = q\int E' \cos \theta' \, dL'.$$

If the electric field is a *static* field, $W = W'$, and therefore

$$\int E \cos \theta \, dL = \int E' \cos \theta' \, dL'.$$

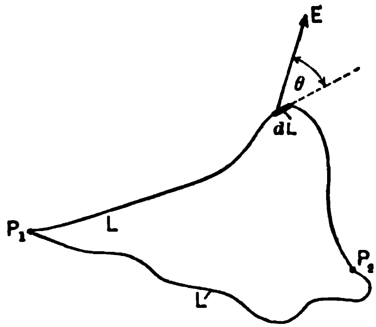


Fig. 2.

For if the work done along any path L were greater than that done along any other path L', a positive amount of work, $W - W'$, would be done on the charged body by the field during each completion of a circuit from P_1 to P_2 along L and back along L', and yet the energy of the field would remain unaltered. Since this is inconsistent with the principle of the conservation of energy, $\int E \cos \theta \, dL$ is the same for every path between two given points in an *electrostatic* field.

The line integral of the electric intensity, $\int E \cos \theta \, dL = W/q$, along a path L from P_1 to P_2 is called the *electromotive force* (e.m.f.) or *voltage* along the path L from P_1 to P_2. When, as in the case just considered, this quantity is the same for every path from P_1 to P_2, it is called also the *difference of potential* between P_1 and P_2, or the *fall of potential* from P_1 to P_2.

Since a voltage is a quantity of work divided by a charge, it is evidently not a vector.

When W is expressed in ergs and q in the *RES* unit charge, or when E is expressed in the *RES* unit intensity, and L in cm., the voltage ($= W/q = \int E \cos \theta \, dL$) is said to be expressed in the *RES unit voltage.* In magnitude, the voltage between two points is equal to the work done in carrying unit charge from

one point to the other along the given path, or any path if the voltage is a potential difference.

18. Potential. Equipotential Surfaces. The fall of potential from a given point P to any point at an infinite distance from all electrified bodies is called the *electric potential* at P.

This term is also commonly applied to the fall of potential from P to any point of the *earth*. That the two definitions are not identical will be shown in § 6, Chapter II.

The symbol V will be used to denote the potential at a point P. In conformity with this notation, the fall of potential from a point P_1 to a point P_2 will be written $V_1 - V_2$, V_{12}, or, where there is no danger of confusion, simply V.

A surface which is everywhere normal to the electric intensity, and between any two points of which there is therefore no voltage, is called an *equipotential surface*, or simply an *equipotential*. It is clear that an equipotential surface is always a closed surface or else (in certain ideal fields) an infinite plane.

19. Electric Intensity in a Static Field the Space Rate of Diminution of Potential. — For the voltage from P_1 to P_2 we have

$$V_1 - V_2 = \int E \cos\theta \, dL = \int E_L \, dL$$

by writing E_L for $E \cos\theta$, the component of electric intensity in the direction of dL. That is, the potential of P_1 exceeds that at P_2 by $\int E_L \, dL$ from P_1 to P_2; or the diminution of potential from P_1 to P_2 is $\int E_L \, dL$ from P_1 to P_2. If the two points are taken an infinitesimal distance dL apart, the diminution of potential along dL becomes $-dV$, and the integral becomes simply $E_L \, dL$. Thus we have

$$-dV = E_L \, dL$$

and therefore

$$E_L = -dV/dL \tag{6}$$

That is, the component of electric intensity in any direction is the space rate of diminution of the electric potential in that direction.

V obviously diminishes most rapidly along a line of intensity, and not at all along a line in an equipotential surface.

20. Electric Field Mapped out by a System of Equipotentials. If a line of intensity is denoted by N, the last equation gives

$$E_N = E = -\,dV/dN.$$

From this it follows that an electric field can be completely mapped out by a system of equipotential surfaces so drawn that the voltage between successive surfaces is constant. For the direction of the intensity at any point is that of the normal to the equipotential passing through the point; and its magnitude is, by the above equation, proportional to the number of successive equipotential surfaces crossed at the point per unit length by this normal or line of intensity. Maxwell's method of drawing such an equipotential system is described in §§ 7, 11, 13, 14, II.

21. Electric Flux. Let dS, Fig. 3, denote an element of area at any point of an electric field where the displacement is D, and let the angle between D and the normal N to dS be denoted by θ. The product of dS into the component of D normal to dS, that is, $D\cos\theta\ dS$, is called the *electric flux* across dS.

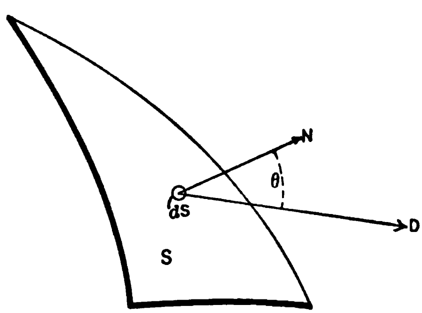


Fig. 3.

To obtain the electric flux, Π, across an extended surface S, over which D may vary in any manner, the integral of $D\cos\theta\ dS$ must be taken over the whole surface. Thus

$$\Pi = \int D\cos\theta\, dS \tag{7}$$

22. Gauss's Theorem: The electric flux outward across any closed surface S so drawn as to enclose a total charge q is equal to q.

The theorem will first be established for the case in which all space is filled up with a single homogeneous isotropic dielectric with permittivity c (or with any number of isotropic dielectrics all of which have the same permittivity c).

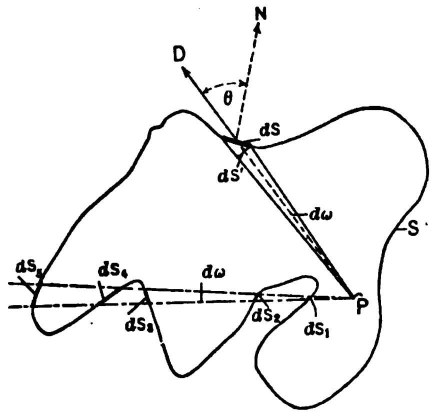


Fig. 4.

Consider first the field about a charge q concentrated at P, Fig. 4, any point within S, a closed surface of any shape. For the magnitude of the displacement, D, at any element of area dS, distant L from P, we have from (1), (2), and (3)

$$D = cE = c(q/4\pi cL^2) = q/4\pi L^2$$

In direction, D and E are evidently radial from P (or to P if q is negative).

For the flux across dS we have therefore

$$d\Pi = D\cos\theta\, dS = q\, dS \cos\theta/4\pi L^2 = q\, dS'/4\pi L^2 = q/4\pi\cdot d\omega$$

where $dS' = dS\cos\theta$ is the projection of dS normal to L, and $d\omega = dS'/L^2$ is the elementary solid angle subtended at P by dS and dS', that is, the angle of the elementary conical tube of induction cutting out the area dS.

If the surface is folded, so that some of the tubes cut it more than once, as the tube of angle $d\omega$ which cuts out the areas dS_1,

$dS_2, \cdots, dS_5$ in the figure, each of these tubes must obviously cut it an odd number of times. And since the angle $d\omega$ of the cone is the same for all the elements dS_1, dS_2, etc., the magnitude of the flux across each will be the same, viz., $q/4\pi \cdot d\omega$, but the flux will be outward (positive) across all the elements with odd numbers, and inward (negative) across all the elements with even numbers. Thus all the elements except one, across which the flux is positive or outward, cut one another out in pairs, leaving the total flux outward through the tube equal, as for a tube of the same angle cutting the surface but once, to $q/4\pi \cdot d\omega$.

The outward flux across the complete surface is therefore

$$\Pi = \int d\Pi = q/4\pi \int d\omega = q \tag{8}$$

since the whole solid angle, $\int d\omega$, subtended by any closed surface at a point within it is 4π.

This result is independent of the position of P within S; hence, by the principle of superposition, it must hold for charges distributed in any manner within S, q denoting now the total (algebraic) charge within. The validity of the theorem for all isotropic electrostatic fields will be established later (§§ 29, I. and 1, IV.).

23. The Strength of a Tube of Induction. From (8) it follows that the flux across every cross-section of a given tube of induction is the same. For by the definition of a tube there is

Fig. 5.

no flux across any part of its sides; and since in the space enclosed within the sides and two diaphragms S_1 and S_2, Fig. 5, there is no electric charge, the flux which enters this region

across S_1 must equal that which leaves across S_2. Thus there is an analogy between the electric flux and the flux of an incompressible fluid.

The *strength* of a tube of induction is defined as the magnitude of the flux across any diaphragm of the tube. A unit tube is a tube whose strength is unity.

24. Electric Charge and Discontinuity of Electric Flux. With the exception of closed tubes of induction (Chapter VI.), all tubes in a static field emanate from positively charged bodies and terminate upon negatively charged bodies. To prove this statement, consider two electrified bodies (there cannot be less than two) alone in the field, there being no charges upon other bodies. If one possesses the charge $+q$, the other possesses the charge $-q$ (§ 6). The total electric flux outward across any closed surface surrounding $+q$ is q, and the total inward flux across any closed surface surrounding $-q$ is q; or the total flux across any closed surface separating the charge $+q$ from the charge $-q$ is equal to q in magnitude, and in direction is from $+q$ toward $-q$. That is, all the tubes emanate from the body with charge $+q$ and terminate upon that with charge $-q$, the total strength of all the tubes being q.

Exactly the same mode of reasoning may be applied to a single tube of induction. The strength of a tube is thus equal to the magnitude of the positive charge at one end or to the magnitude of the negative charge at the other. The whole electric field indeed may be regarded as a single tube of induction passing from one charge to the other.

Thus the electric charge resides only where the displacement is discontinuous, and is measured by the amount of this discontinuity. In fact Gauss's theorem simply states the identity of an electric charge and the flux from the charge, or rather the discontinuity of the flux at the charge.

Rational Units. The system of units here adopted is called rational for the reason that it makes the flux from a charge equal

to the charge numerically, as it is dimensionally, instead of to 4π $\times$ the charge, as in the common systems, and, as a consequence, does away with the factor π except in the case of spherical or circular distributions, where it would naturally occur.

25. Electric Field Mapped Out by Tubes of Induction. In the elementary tube T, Fig. 6, let the diaphragms dS_1, dS_2, be drawn

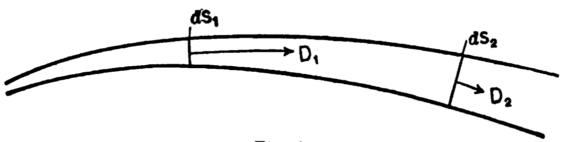


Fig. 6.

at right angles to the axis of the tube. Then we have, by §23, $D_1dS_1 = D_2dS_2$, whence

$$D_1/D_2 = dS_2/dS_1 = E_1/E_2 \tag{9}$$

Thus the intensity and induction at every point along a narrow tube are inversely proportional to its right cross-section at the point. Since therefore the magnitude of the right cross-section of a tube at a point indicates the magnitude of the induction and intensity, and the direction of the tube the direction of these quantities, an electric field may be completely mapped out by drawing a system of tubes, all of the same strength, filling the field. Maxwell's method of drawing such a system of tubes will be explained in §§ 7, 11, 13–14, II.

26. The Surface of a Conductor in a Static Field is an Equipotential Surface. For, since in a static field there is no electric intensity within the substance of a conductor, the voltage $\int E \cos \theta \, dL$ is zero along any line drawn wholly through the substance of a conductor and connecting any two points of its surface (and therefore along any other line connecting the two points, since the field is static).

27. Equipotential Region. If in the region on one side of a given equipotential surface there is no electric charge, the elec-

tric induction and intensity in this region are also zero, and all parts of it are therefore at the same potential as that of the equipotential surface. For all the tubes which cross an equipotential surface cross it normally and but once; and in the region considered there is no electric charge with which such tubes could originate or terminate. Hence there are no tubes in the region, by Gauss's theorem, and no voltage.

That the space containing the substance of a *conductor*, or the space included within a hollow closed conductor, is an equipotential region, has already been established.

28. In a Static Field a Conductor may be Replaced by a Dielectric of any Permittivity. Since there is no electric field in a region without charge bounded by an equipotential surface (charged or uncharged), the substance filling this region may be replaced by any other substance, with its surface charged in the same manner as that of the substance replaced, without in any way affecting the electric field. Thus it is extremely convenient for the purpose of solving many electric problems, to imagine the substance of an electrified *conductor* replaced by a dielectric of the same permittivity as that of the surrounding medium, with its surface coincident with that of the conductor and charged in the same manner. This is in order to apply the law of inverse squares, which can be done only when all space contains the same dielectric of uniform permittivity. Extensive use will be made of this principle in what follows and it will be generalised in Chapter IV.

29. Gauss's Theorem Valid for a Finite Region and for a Field Containing or Bounded by Conductors. As an immediate corollary of what precedes, it follows that Gauss's theorem is valid throughout an infinite electric field containing a homogeneous isotropic dielectric and any number of conductors. And as an immediate corollary of this last proposition and § 4, it follows that the theorem is valid throughout any finite electric field

bounded by conductors, and throughout a finite portion of any electric field, provided that this finite field or portion of a field contains only a single homogeneous isotropic dielectric and conductors. The validity of the theorem is still further extended in Chapter IV.

30. Electric Surface and Volume Density. Convergence and Divergence of a Vector. The *electric surface density* at any point of a charged surface is defined as the charge per unit area at the point, and will be denoted by σ. If dS is an element of area at the point and dq its charge,

$$\sigma = dq/dS \tag{10}$$

The outward flux across any surface enclosing dq and no other electric charges is $d\Pi = dq = \sigma dS$, by Gauss's theorem (not yet proved for this case, since the surface encloses, in general, two dielectrics). Let such a surface be formed by a right cylinder of infinitesimal length drawn through the

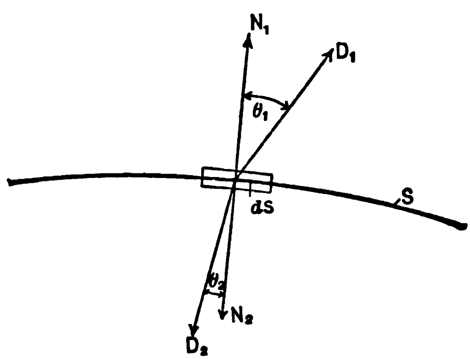


FIG. 7.

boundary of dS and closed up by two planes parallel with dS, one on each side, Fig. 7. The lateral area of this cylinder is negligible in comparison with that of the ends, so that the outward flux across the total surface is equal to the flux across the ends. Therefore, if D_1 and D_2 are the displacements on the two

sides of dS, and θ_1 and θ_2 the angles they make with the normals drawn from dS,

$$d\Pi = \sigma dS = (D_1 \cos\theta_1 + D_2 \cos\theta_2)dS$$

whence

$$\sigma = D_1 \cos\theta_1 + D_2 \cos\theta_2 = c_1 E_1 \cos\theta_1 + c_2 E_2 \cos\theta_2 \qquad (11)$$

if c_1 and c_2 denote the permittivities of the media on the two sides of dS.

If the charged surface is that of a *conductor* in a static field, the displacement, D, on one side is normal to the surface, and on the other side is zero; so that in this case (11) becomes

$$\sigma = D = cE \qquad (12)$$

which might have been written down at once from Gauss's theorem, already established for this case.

The *electric volume density* at any point of an electrified volume is defined as the charge per unit volume at the point, and will be denoted by ρ. If dq is the charge in the element of volume $d\tau$ at the point,

$$\rho = dq/d\tau \qquad (13)$$

The electric flux outward from dq through the surface of $d\tau$ is $d\Pi = dq = \rho d\tau$, whence

$$\rho = dq/d\tau = d\Pi/d\tau = \text{div } D \qquad (14)$$

The symbol *div* D is an abbreviation for the *divergence* of D, which is another name for $d\Pi/d\tau$, the outward flux of the vector D per unit volume, or, in magnitude, the amount of the flux leaving unit volume through part of its surface minus the amount entering the same volume through the rest of its surface.

If ρ is negative, div D is also negative, or the flux is, on the whole, directed *into* $d\tau$. To the negative of the divergence the term *convergence* is applied. It is written *conv.* Hence

$$-\rho = -\text{div } D = \text{conv } D \qquad (15)$$

The convergence or divergence of any other vector is similarly defined as the inward or outward flux of the vector per unit volume at the given point.

An insulator may possess both volume density and surface density of electrification, but the charge of a conductor in a static field resides, as has been already shown, on the surface only. This statement must not be taken too literally, however, as the molecular structure of matter makes it necessary that the displacement should terminate upon the atoms of a *surface layer*, although this layer is extremely thin.

31. Cartesian Expression for the Divergence and Convergence of a Vector. The Equations of Poisson and Laplace. First we shall obtain the expression for the divergence of the vector D. Let the components of D at the point whose coördinates are x, y, z, parallel to the rectangular axes X, Y, Z, be D_1, D_2, D_3, respectively, Fig. 8. Consider the elementary parallelepiped whose edges are parallel to the coördinate axes and have the infinitesimal lengths dx, dy, dz, the coördinates of the corner nearest the origin of coördinates being x, y, z.

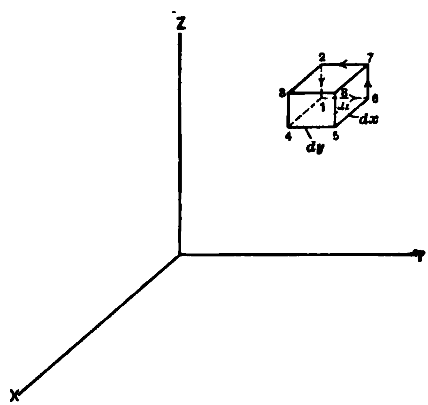


Fig. 8.

The flux *into* the parallelepiped through the face 13 is $D_2\ dxdz$ (or the flux *out* across the face 13 is $-D_2\ dxdz$), and that out through the opposite face, 57, is $(D_2 + dD_2/dy\ dy)\ dxdz$. Hence the resultant *outward* flux across the two faces parallel to the XZ plane is $(D_2 + dD_2/dy\ dy)dxdz - D_2dxdz = dD_2/dy\ dxdydz$.

In exactly the same way the resultant outward flux across the two faces 26 and 35 is $dD_1/dx\ dxdydz$, and that across the faces 46 and 37, $dD_3/dz\ dxdydz$.

Hence the total flux outward from the parallelepiped is

$$d\Pi = (dD_1/dx + dD_2/dy + dD_3/dz)\, dxdydz$$
$$= (dD_1/dx + dD_2/dy + dD_3/dz)\, d\tau$$

and

$$\rho = \text{div } D = -\text{ conv } D = d\Pi/d\tau \qquad (16)$$
$$= dD_1/dx + dD_2/dy + dD_3/dz$$

Similarly, for any other vector, as E, we have

$$\text{div } E = -\text{ conv } E = dE_1/dx + dE_2/dy + dE_3/dz$$

If c is constant (independent of x, y, z), we have, since $D = cE$,

$$\text{div } D = dD_1/dx + dD_2/dy + dD_3/dz \qquad (17)$$
$$= c(dE_1/dx + dE_2/dy + dE_3/dz) = c \text{ div } E$$

Equation (17) may be written

$$\rho = \text{div } D = d/dx\,(cE_1) + d/dy\,(cE_2) + d/dz\,(cE_3) \qquad (18)$$
$$= -\,d/dx\,(cdV/dx) - d/dy\,(cdV/dy) - d/dz\,(cdV/dz)$$

If c is independent of the coördinates, this equation becomes

$$\rho = \text{div } D = -\,c\,(d^2V/dx^2 + d^2V/dy^2 + d^2V/dz^2) \qquad (19)$$

(18) and (19) are the *equations of Poisson*. When $\rho = 0$, the equations become

$$d/dx\,(cdV/dx) + d/dy\,(cdV/dy) + d/dz\,(cdV/dz) = 0 \qquad (20)$$

and

$$d^2V/dx^2 + d^2V/dy^2 + d^2V/dz^2 = 0 \qquad (21)$$

which are the *equations of Laplace*.

32. The Equilibrium of Superposed Electric Fields. (1) If in each of any number of electric fields separately each of a given system of surfaces of fixed configuration is an equipotential, then in the electric field resulting from the geometric superposition of these fields each surface will remain an equipotential. For since in each field separately the tubes meet the surfaces normally, by

definition of an equipotential, the tubes in the geometrically obtained resultant field will also meet the surfaces normally.

(2) That the superposition of any number of distributions of electric charges in or upon insulators gives a resultant distribution of charges in equilibrium is evident from the definition of an insulator. That the resultant field (obtained by geometrical superposition) connected with these charges is in equilibrium, and that this is the only possible resultant field in equilibrium, follows from §12.

(3) If each or any of the equipotentials of (1) encloses no charges, then it encloses no field, and it is immaterial so far as the external (*i. e.*, the only) field is concerned whether the substance within this surface is an insulator or a conductor (§28). If the field is in equilibrium in the one case, it will be in equilibrium in the other. Hence we may state that if each of any number of electric fields surrounding or bounding a given system of conductors with fixed configuration is separately in equilibrium, then the electric field resulting from their geometric superposition will also be in equilibrium, and will be the only possible resultant field in equilibrium (*i. e.*, static). The last statement is proved again in § 46.

33. The Superposition of Voltages and Potentials. If the voltage from any point P to any other point P' is V_1 when the field surrounding the points is a given field A_1, V_2 when the field is A_2, $\cdots$, V_n when the field is A_n, then the voltage from P to P' when all the fields are superposed is

$$V = V_1 + V_2 + \cdots + V_n \tag{22}$$

For, all the integrals being taken along the same path L (which may be *any* path from P to P'), we have, in the notation of §§ 17–19,

$$V_1 = \int E_1 \cos\theta_1 dL,\ V_2 = \int E_2 \cos\theta_2 dL,\ \cdots,\ V_n = \int E_n \cos\theta_n dL$$

and $$V = \int E \cos\theta\, dL$$

By the principle of superposition of electric intensities

$$E \cos\theta = E_1 \cos\theta_1 + E_2 \cos\theta_2 + \cdots + E_n \cos\theta_n$$

Hence

$$V = \int E \cos\theta \, dL = V_1 + V_2 + \cdots + V_n$$

which is identical with (22).

If P' is any point in the region of zero potential, V and V_1, $V_2, \cdots, V_n$ denote the resultant and component *potentials*, respectively at P.

34. Voltages and Charges Proportional. It is clear from §§ 32 and 33 that when the intensity at every point, and therefore the voltage between every two points, of a static electric field is altered in any ratio, the resulting electric field will be in equilibrium, and the electric surface or volume density at every element of charged surface or volume will be altered in the same ratio, and vice versa. The original field has simply been superposed on itself a given number of times.

35. Capacity of an Electrical System. Permittance of a Dielectric. S is Proportional to c. In an electric field terminated by two conductors A and B all the tubes emanate from one of the conductors and terminate upon the other, so that the charges of A and B are equal and opposite whatever their common magnitude, q. This relation still holds when any number of other conductors, uncharged except by induction, are in the field, the tubes connecting A and B simply being rendered discontinuous at the surfaces of these conductors (§ 42). By the last article, if the voltage V_{12} between A and B is altered in any ratio, q will be altered in the same ratio, and vice versa. That is

$$q = SV_{12} \tag{23}$$

where S is a constant, called the *capacity* of the *system AB*, or much better, the *capacity* or *permittance* of the *dielectric bounded by A and B.*

The above equation may be written

$$S = q/V_{12} = \int DdS / \int EdL \tag{24}$$

dS being the element of area of any equipotential surface (necessarily closed around one of the conductors or else extending to infinity), and dL the element of length of any line of intensity, the integrals extending over the whole surface and along the whole length of the line, respectively.

The term permittance is applied to S (surface integral of displacement/line integral of intensity) for the same reason for which the term permittivity is applied to c (displacement/intensity).

When the charge and voltage are expressed in *RES* units in (23) and (24), S is said to be expressed in the *RES unit capacity* or *permittance.* The unit capacity is thus the capacity of a system, or dielectric, upon each of whose terminating conductors the charge is unity when the voltage is unity.

Two conductors which completely bound an electric field, like the system AB, are called, with the intervening dielectric, an *electric condenser* or *leyden.* These terms are commonly applied, however, only when the conductors are near together, in which case the displacement may be very great, or the electric charge highly "condensed" even when the voltage

$$\left(\int EdL = 1/c \int DdL\right)$$

is small (since L is small). The term condenser is also applied to a system in which *nearly* all the tubes of induction pass from one of the conductors to the other.

If the voltage V of a leyden is kept constant, and the permittivity c of its dielectric altered everywhere in a given ratio, the intensity E will remain constant, but the displacement D, and therefore the charge q, will be altered everywhere in the same ratio. Hence S is proportional to c.

Although a charge of one sign cannot exist without the complementary charge of opposite sign, it is sometimes convenient

to *imagine* one of the charges removed to an infinite distance, when the electric field within a finite distance is connected with only a single electrified body and conductors with induced charges. The intensity and potential at every point will then be proportional to the charge of the electrified body.

36. Mechanical Analogue of the Relation $q = SV_{12}$. If a spring (analogous to the dielectric of a condenser) which obeys Hooke's law and has perfect elasticity ($c = D/E =$ constant) is stretched a distance L (analogous to q) by a force F (analogous to V_{12}) then

$$L = KF \tag{25}$$

where K (analogous to S) is a constant depending on the spring.

A similar relation of course exists between the deformation and the forcive in the case of any other perfectly elastic strain.

37. The Electrostatic Energy of a Field Bounded by Two Conductors. Energy Contained in a Tube of Displacement Between Two Equipotentials. The energy contained in the dielectric bounded by the two conductors A and B due to its electric displacement is equal to the work done in creating the electric field, or the work done against the electric field in charging the system (provided there is no dissipation of energy in dielectric hysteresis, § 1, VI.). Let the process of charging consist in carrying successive elements of charge dq from B to A, or $-dq$ from A to B, or both. Each time this is done A gains a charge $+dq$ and B a charge $-dq$, and the work done in effecting the transfer, if the charges of A and B, at the time are $+q$ and $-q$ respectively, and if the corresponding voltage from A to B is V, is

$$dW = Vdq \tag{26}$$

by (5).

If $S = q/V =$ constant, which is true except when intrinsic displacement (VI.) is present, (26) may be written

$$dW = Vdq = 1/S\, qdq = SVdV$$

Hence the total work done in establishing the field, or the total electrostatic energy of the field, is

$$W = \int_0^q V dq = 1/S \int_0^q q dq = \tfrac{1}{2} q^2/S$$
$$= S \int_0^V V dV = \tfrac{1}{2} S V^2 = \tfrac{1}{2} q V \qquad (27)$$

The electric field may be considered as a single tube of displacement connecting A and B, the strength of the tube being q and its voltage V. The energy of this tube is then one half the product of its strength by its voltage. Or the field may be divided up into tubes of displacement in any manner, and since the above result is wholly independent of the shapes of the tubes, the energy contained in each tube is in the same way one half of the product of its strength by its voltage. Also, the energy contained in the portion of any tube of strength q between two equipotential surfaces differing in potential by V is $\frac{1}{2} q V$, whether the tube terminates at these surfaces or not.

In any case, whether energy is dissipated or not, or whether $q/V = S$ is constant or not, the work done in charging the condenser from a neutral state to charge q, or the work done in changing the strength of a tube of displacement from O to q, and its voltage from O to V, is

$$W = \int_0^q V dq \qquad (28)$$

38. Electric Energy Density in a Dielectric. From § 37 it follows that when $D = cE$ (no intrinsic electrisation present, Chapter VI.) the energy per unit volume at any point of an electric field is

$$U = \tfrac{1}{2} ED = \tfrac{1}{2} c E^2 \qquad (29)$$

To prove this, consider an elementary tube, of strength dq, cutting two equipotential surfaces distant dL apart, the point considered being at the center of the element of volume $d\tau$ enclosed by the sides of the tube and the equipotentials. If the right

cross-section of the tube is dS at the point, $d\tau = dL\,dS$. The energy contained in $d\tau$ is

$$dW = \tfrac{1}{2}dq\,EdL = \tfrac{1}{2}DdS\,EdL = \tfrac{1}{2}ED\,d\tau = \tfrac{1}{2}cE^2\,d\tau$$

and the energy per unit volume is

$$U = dW/d\tau = \tfrac{1}{2}ED = \tfrac{1}{2}\,cE^2$$

which is identical with (30).

Without assuming the relation $q/V = S =$ constant, or $c = D/E =$ constant, or that there is no dissipation of energy, we can show that the work done per unit volume in creating a displacement D is

$$U = \int_0^D EdD \qquad (30)$$

which reduces to (29) when $c = D/E =$ constant.

For in the general case, § 37 (28), the work done in changing the strength of a tube of displacement from O to q is

$$W = \int Vdq = \int\int\int EdDdLdS = \int\int EdD\,d\tau$$

which, on differentiating with respect to τ, gives (30).

39. Electric Tension and Pressure (Preliminary). From the consideration of a static electric field (such as the field of Fig. 22, 24, or 47), in which tubes of induction stretch, in general, from a positively charged body to another body negatively charged; in which there is always a force of attraction between the oppositely charged bodies; and in which a small electrified body (if the force of gravity is eliminated) will move along a line of intensity; it follows immediately that at every point of an electric field there is a tension in the dielectric in a direction parallel to the intensity—the tubes of induction tending to contract in length indefinitely and to pull together the electrified bodies on which they end.

It is clear also from the manner in which the tubes of induction spread out laterally as they pass from one of the bodies to the

other, filling all space except as the field is bounded by conductors, that at every point in the dielectric there is a pressure perpendicular in every direction to the intensity at the point. Were the tension along the tubes the only stress, it is clear that *all* the tubes would contract in cross-section as well as in length and stretch *straight* across from one charge to the other; and the electromotive force from charge to charge along all paths not passing through the region occupied by these tubes would be zero, which is of course impossible.

It is clear also that the intensity and pressure at any point are greater the greater the intensity and induction at the point. These stresses are referred to in § 14, and will receive detailed consideration in the next two articles.

40. Electric Tension, Method I. At any point in a dielectric in which (29) holds there is a tension in the direction of the intensity, with magnitude per unit area

$$T = \tfrac{1}{2}ED = \tfrac{1}{2}cE^2 = U \tag{31}$$

To prove this, consider a uniform field which is terminated at one end by a plane conducting plate of area A (necessarily) normal to the electric field (III, § 2). If the plate is moved in a direction parallel to the field an infinitesimal distance dL, the volume of the dielectric under strain terminated by the plate of area A is increased by AdL and the energy by $dW = AdL\,\tfrac{1}{2}ED$, the displacement of the plate being so small that E and D remain sensibly unaltered. This increase in energy is equal to the work done in moving the plate the distance dL against the force normal to its surface due to the tension in the dielectric. If the force per unit area on the plate, which must equal the tension in the direction of the intensity in the dielectric, is denoted by T, we have therefore

$$dW = TAdL = \tfrac{1}{2}ED\,AdL;\ \text{and}\ T = 1/A\,dW/dL = \tfrac{1}{2}ED,\ \text{etc.},$$

which is identical with (31).

This result has been deduced for a uniform field, but since every field is uniform throughout an infinitesimal volume, the result is perfectly general.

The best form of apparatus for investigating the electric tension experimentally is described in §§ 2 and 4, III.

Electric Tension, Method II. The proposition just established may also be demonstrated as follows: Let dS be an element of the charged surface of a conductor, and let P and Q be two points indefinitely near the surface, one without and the other within the conductor at the center of dS. Consider the substance of the conductor replaced by a dielectric of the same permittivity as that of the surrounding medium (c), § 28. Then the electric intensity at P and at Q may be resolved into two components, one which can be calculated from the charge upon dS, and the other from the rest of the charges in the field (or, as ordinarily expressed, one *due* to the charge on dS, and the other *due* to the other charges). Let the two components at P be denoted by E_1 and E_2, and the resultant intensity by E. By symmetry, E_1 is normal to dS, and there is an equal and opposite component, $-E_1$, at Q. Since E and E_1 are both normal to dS, E_2, their vector difference, is also normal. Hence

$$E = E_1 + E_2$$

The resultant intensity at Q, inside the surface, is zero, and has the components, E_2 normally outward, and E_1 normally inward (that is, $-E_1$). Hence

$$O = E_2 - E_1$$

Therefore $$E_1 = E_2 = \tfrac{1}{2}E$$

The charge upon the element of surface dS is $\sigma dS = DdS$; and, since the intensity at the charged element due to σdS is zero (being directed symmetrically toward the outside and inside), and since the intensity due to the other charges is $\frac{1}{2}E$, the mechanical force per unit area upon the charged surface between P and Q is

$$T = \tfrac{1}{2}ED = \tfrac{1}{2}\sigma^2/c = \text{etc.} \qquad (31)$$

Electric Tension, Method III. The same result may be obtained by still another method. As we have seen, the electric charge is not strictly a surface distribution, but is confined to a very thin surface layer. At the outer surface of the layer

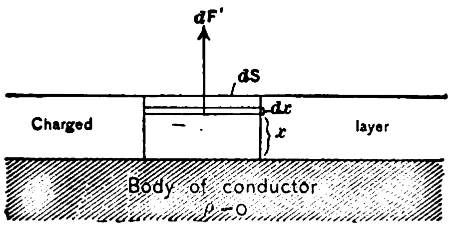


Fig. 9.

the intensity and displacement, which are normal to the surface throughout the layer, have their full surface values E and D; at the inner boundary of the layer they are zero. If E and D denote also the intensity and displacement at a distance x from the inner boundary of the surface layer, of thickness L (Fig. 9), the charge within the small volume of thickness dx and cross-section dS is

$$dq = dx\, dS \operatorname{div} D = dx\, dS c dE/dx$$

E being a function of x only. The outward force upon the portion of the conductor within this charged volume is

$$dF' = E dq = dS\ cEdE/dx\, dx$$

and the total force upon that part of the surface layer whose cross-section is dS is

$$dF = \int dF' = dS\int_0^L cE\, dE/dx\, dx = dS\, c\int_0^E EdE = dS\, \tfrac{1}{2}cE^2$$

whence the force per unit area upon the charged surface, or the tension in the dielectric at the surface, is

$$T = dF/dS = \tfrac{1}{2}cE^2 = \tfrac{1}{2}ED \qquad (31)$$

41. Electric Pressure. Equilibrium of a Dielectric Supporting Electric Displacement. In a dielectric supporting an electric

field there is at every point, in addition to the tension $\frac{1}{2}ED$ in the direction of the intensity, a pressure normal in every direction to this intensity and equal to

$$p = \tfrac{1}{2}ED = T = U \tag{32}$$

To establish this proposition, consider the radial field from a charge upon a very small body at P, Fig. 10, and an elementary (conical) tube of displacement T cutting two (spherical) equipo-

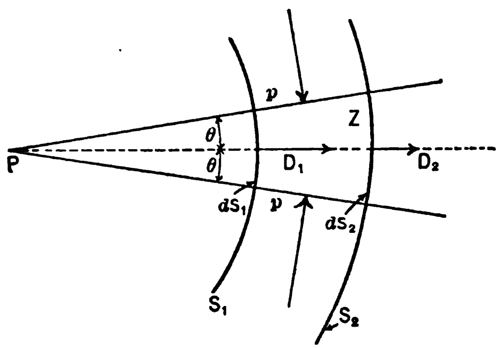


Fig. 10.

tentials S_1 and S_2 a distance dL apart, and enclosing areas dS_1 and dS_2 of these surfaces. Let E_1, D_1 and E_2, D_2 be the intensity and displacement at S_1 and S_2, respectively. The portion Z of the dielectric enclosed by the sides of the tube and S_1 and S_2 is in equilibrium under the action of the stresses of the field. The force on Z arising from the tensions is, by (31),

$$\tfrac{1}{2}cE_1^2 dS_1 - \tfrac{1}{2}cE_2^2 dS_2$$

measured toward P, and must be balanced by an equal force directed from P. We shall assume that this equilibrating force arises from a pressure p normal everywhere to the surface of the tube, and shall proceed to find its value. If p_1 and p_2 are the values of p at dS_1 and dS_2, respectively, its average value over the surface of the small volume Z is approximately

$$\tfrac{1}{2}(p_1 + p_2)$$

From the figure it is clear that the resultant force due to p is outward along the axis of the tube and equal, approximately, to

$$\tfrac{1}{2}(p_1 + p_2) \sin \theta \, B$$

if B is the lateral area of the surface of Z. Since

$$\sin \theta = (dS_2 - dS_1)/B$$

very approximately,

$$\tfrac{1}{2}(p_1 + p_2) \sin \theta \, B = \tfrac{1}{2} (p_1 + p_2)(dS_2 - dS_1)$$

Z is evidently, by symmetry, in equilibrium laterally, so that it will be in complete equilibrium if

$$\tfrac{1}{2}(p_1 + p_2) (dS_2 - dS_1) = \tfrac{1}{2}cE_1^2 dS_1 - \tfrac{1}{2}cE_2^2 dS_2$$

Since

$$E_1 dS_1 = E_2 dS_2$$

the last equation may be written

$$\tfrac{1}{2}(p_1 + p_2) (dS_2 - dS_1) = \tfrac{1}{2} cE_1 E_2 (dS_2 - dS_1)$$

which becomes, when the tube T is made indefinitely narrow, and the surfaces S_1 and S_2 are brought indefinitely close together,

$$p = \tfrac{1}{2}cE^2 = \tfrac{1}{2}ED$$

which is identical with (32).

Since the field within the element of volume is uniform when the element is made indefinitely small, and since this is true of any electric field, the result just obtained for a radial field holds universally.

A method of proving (32) by direct experiment is described in § 3, VII.

42. Electric Conduction and Induction. The tension in the direction of the intensity at every point of a dielectric supporting an electric field and the pressure perpendicular to this direction throw much light on the disappearance or transfer of electric charges by conduction and their development by induction.

Conduction. First it will be shown that this stress system will account for the result of § 26. For if the surface of a conductor were not an equipotential, *i. e.*, if the tubes of induction did not meet the surface normally, there would be in the dielectric at the interface a component of the intensity parallel to the surface, and therefore a component of the tension parallel to the surface and a component of the pressure perpendicular to the surface. Since the charges, or the ends of the tubes, can move freely along a conductor, the tubes would therefore contract, their ends, or the charges, slipping along the conductor; and since within the substance of the conductor the intensity, and therefore the pressure perpendicular to the surface, is zero, the component perpendicular to the interface of the pressure in the dielectric would be unbalanced by any pressure from within, so that the tubes would be continually pushed toward and into the conductor (there to break up). Since these processes are inconsistent with the nature of a static field, there can be no component of the intensity parallel to the surface of the conductor.

Consider two electrified conductors A and B with positive and negative charges respectively, A being the only positively charged body in the field; and suppose q, the charge of A, numerically greater than q', the charge of B. Of the q unit tubes emanating from A, q' terminate upon B, and $q - q'$ upon other bodies at a distance. If A and B are connected by a wire C, in which permanent electric stress cannot exist, the tensions along the tubes and pressures at right angles to them will cause the q' tubes connecting A and B to be pushed, contracting as they go, into the regions of no permanent stress, the conductors A, B, and C, until the positive and negative ends of each tube meet and the tube disappears.

The remaining $q - q'$ tubes emanating from A will be redistributed by the system of tensions and pressures until there is again equilibrium, when the remaining $q - q'$ tubes will emanate normally from A, B, and C.

During the process of conduction the field is not in equilibrium, nor is it zero within the conductors, and the tubes are not normal to the surfaces of the conductors, but are inclined from the normal at each end in the direction of motion of that end. If the conductivity were perfect, the tubes would always end normally at the conducting surfaces and would never disappear in the conductors (Chapter VIII., § 9).

The phenomena here described are only a part of the phenomena occurring during conduction, and a more complete discussion will be given later (Chapters VIII., XII.).

Induction. Into an electric field, as that bounded by a concentrated charge A and the walls of the room, let a conductor B be introduced. The state of strain previously existing in the space now occupied by B is annulled by its introduction, the tubes formerly crossing this space being cut in two by the conductor, and those sufficiently near being pushed against its surface and there also cut in two, until all the tubes so severed touch B normally and there is again equilibrium. For every tube terminating upon B there is therefore a tube of equal strength emanating from it. That is, the positive and negative charges developed by induction are equal.

All the tubes severed by B may be regarded as still belonging to A; they are simply rendered discontinuous at the surface of B, where the induced charges therefore reside.

If B is connected to the walls by a wire C, the tubes stretching from B to the walls will disappear by the process of conduction, described above, leaving B charged oppositely to A. The tubes between B and the walls having disappeared, more of the tubes from A will crowd into their places until there is again equilibrium, part to end on B, part on C, and part on the walls. The charge on B of the opposite kind to that of A is thus increased by earthing B.

Fig. 11 (from Nichols and Franklin's *Elements of Physics*, Vol. II., §§ 165–6) illustrates the process of introducing a small

charged conductor B, insulated, into a nearly closed hollow conductor A, §4, putting on the conducting lid, moving B about inside, and finally bringing B into contact with A's inner surface. The distribution of the tubes here, as well as in the preceding cases, can be roughly predicted from the considerations that all the tubes meet both conductors normally and that the voltage

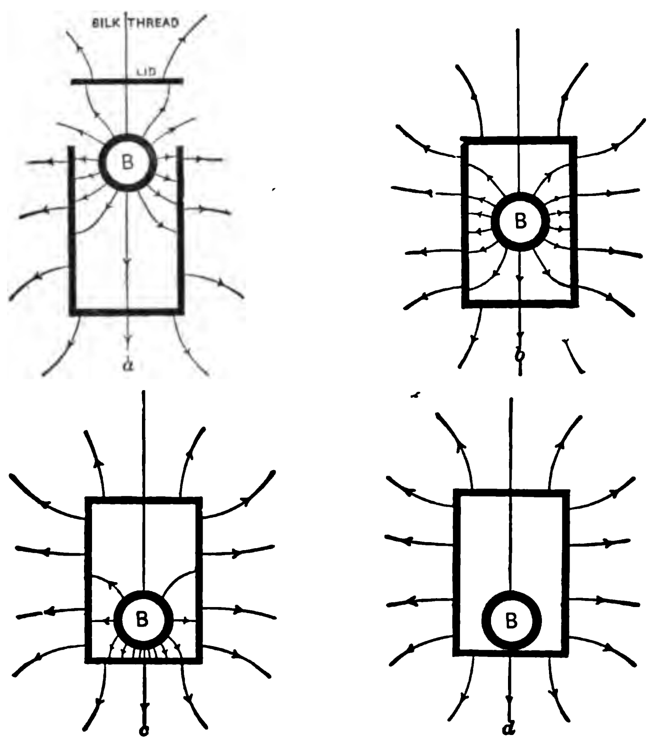


Fig. 11.

along every line from one conductor to the other is the same. Since the voltage along a line of intensity is equal to the average value of the intensity along the line × its length, the intensity and the induction must be greater, or the tubes more concentrated, the shorter the distances through which they stretch.

If, instead of a conductor, an insulator of permittivity different from that of the dielectric in the region is introduced into the

field, phenomena similar in some respects are observed. This subject will receive consideration in Chapter IV.

43. Electric Tension and Pressure and Forcives in the Electric Field. The force upon a *conductor* in or bounding a static field is due wholly to the tension in the dielectric, the force per unit area at any point of the surface being T normal to the surface. Since E and D are perpendicular to a conductor's surface, no component of the force upon a conductor can be due to the electric pressure, which is tangential to the surface and balanced in every direction. Inasmuch, however, as the *distribution* of the tubes, and thus the distribution of T, over the surface of a conductor is determined by both tensions and pressures, the latter contribute to the force indirectly.

The force upon a *dielectric* is in general due to both tensions and pressures. See Chapter IV., § 9.

In Chapers II., III., IV., VI. and VII. many examples will be found.

44. The Equilibrium of a Given Field is not Altered if its Direction is Reversed at Every Point. For the tension parallel to the intensity and the pressure perpendicular to the intensity, at any point of the field are proportional to its square, and are therefore not altered by the reversal of direction. The signs of all charges are of course reversed with the reversal of the intensity.

45. When the Algebraic Charge upon Each Conductor of an Isolated System is Zero, there is no Charge and no Field. If the total algebraic charge of each of a system of conductors is zero, and if there are no other electrified bodies, or electrets (Chapter VI.), or a changing magnetic flux (Chapter XIII.) anywhere, then there is no electric charge or displacement anywhere. For if the electric field were not zero, tubes of displacement would emanate from each conductor and terminate upon conductors of lower potential, since each conductor would have both positive and negative charges. And since the algebraic charge of each

conductor is zero, as many tubes as emanated from the conductor at highest potential would terminate upon its surface, while there would be no body at higher potential at which such tubes could originate; and as many tubes as terminated upon the conductor at lowest potential would emanate from its surface, while there would be no body at lower potential for these tubes to terminate upon. Thus the supposed case is impossible, and there is no charge or intensity anywhere.

46. A Single Electric Field Corresponding to Given Charges. If an electric field A_1 bounded by a system of fixed conductors with given total charges, or by a system of insulators with both charges and their distribution given, or by both, is in equilibrium, this is the only field satisfying the given conditions which is in equilibrium. For suppose that A_2 is a second field satisfying the conditions: It will be shown that $A_2 = A_1$. For if A_2 with its sign reversed is superposed upon A_1, the resultant field will be in equilibrium, and the charge at each point of every insulator and the total charge of each conductor (the last statement in § 32 not being assumed as known) will be zero. Hence by the last article there is no electric charge or displacement anywhere. Thus at every point $D_1 + (-D_2) = 0$, or $A_2 = A_1$.

A Single Field Corresponding to Given Potentials. If an electric field A_1, bounded by a given system of conductors, the potential of each being given, or the voltage between each and all the rest, is in equilibrium, this is the only equilibrium field satisfying the given conditions. For let A_2, another field, supposedly, satisfying the conditions be superposed upon A_1 with its sign reversed. In the resultant field the potential of each conductor, or the voltage between each and all the rest, is zero, by the last article. Hence the intensity is zero everywhere, and $A_2 = A_1$.

47. Equipotential Replaced by Infinitely Thin Conductor of Same Shape. In any electric field any equipotential surface

S (always closed or else an infinite plane) can be replaced by an infinitely thin conducting sheet S' without disturbing the electric field on either side. For all the tubes which before the substitution crossed S normally, after the substitution terminate normally on one side of S' and emanate normally from the other side (induced charges being developed). But since S' is coincident with S, this necessitates no change in the direction or position of any tube, and the substitution therefore leaves the field in undisturbed equilibrium, the tensions and pressures remaining precisely the same as before the substitution. This result is also a corollary of the next article.

Definition of Electric Images. The two charges, or systems of charges, in the regions on opposite sides of S' (but not including the charges upon S') are called the *electric images* of one another in the surface S'.

Since the conducting sheet S' renders the fields on its opposite sides absolutely independent of one another, either field may be destroyed or modified in any manner without affecting the other. If we are concerned with only one of these fields, the substance of the conductor whose surface S' coincides with S may be extended in any manner into the region (previously) occupied by the other field.

48. Additional Propositions Fundamental to the Method of Electric Images. If the potential of a surface S and the charges on one side of S (closed or an infinite plane, being an equipotential) are given, the electric field on this side is fixed independently of the way in which the surface is kept at the given potential. For suppose that two fields A_1 and A_2 satisfy the conditions. If A_2 with its direction everywhere reversed is superposed upon A_1, S will be at zero potential, and there will be no electric charge on the side of S considered. Hence there is no field on this side, and $A_2 = A_1$.

Similarly, if the position of an equipotential surface S is given, together with the flux across it and the charges on one side of it,

the field on this side is fixed and independent of the way in which S is kept equipotential and of the way in which the flux across it is kept of the given magnitude. For suppose that two fields A_1 and A_2 satisfy the conditions. If $-A_2$ is superposed upon A_1, the surface S will still be an equipotential, the flux across it will be algebraically zero, and there will be no charge on the side of S considered. Hence not only the total flux across S is zero, but the flux across every part of S, and there is no field on the side of S considered. Thus, as before, $A_2 = A_1$.

In the above two cases, if the given charges are upon conductors, only the total charges need be given; but if the charges are upon insulators, the distribution as well as the magnitude of each charge must be given.

49. The Electrostatic Energy of an Electric Field Surrounding any Number of Conductors. We shall now proceed to find the energy of an electric field containing any number of conductors, $A_1, A_2, \cdots, A_n$, with any charges $q_1, q_2, \cdots, q_n$, and at any potentials $V_1, V_2, \cdots, V_n$; V_1 and V_n being the highest and lowest potentials, respectively. For the sake of keeping the field within finite limits and eliminating the field of the earth, let all the conductors A_1, A_2, etc., be enclosed within a hollow closed conductor A_o, such as the walls of a room, at potential V_o. This limitation will be removed later for ideal cases. Some of the conductors will, in general, be at higher, and some at lower potentials than V_o. Let the field be divided into regions of higher and lower potential than V_o (by equipotential surfaces of potential V_o).

Consider first the conductors $A_d, A_e, \cdots$, in a region above V_o in potential. On A_d, the conductor at highest potential in the region, no tubes terminate, and from it q_d tubes emanate, some terminating on A_o and others cutting the part of the equipotential V_o separating the region under consideration from the neighboring region at lower potential. Some of the q_d tubes are discontinuous at the surfaces of the conductors A_e, A_f, etc., in the region

(the discontinuities corresponding to induced charges), but all finally reach the boundary, at potential V_o, of the region. The voltage along every tube from A_d to the boundary of the region is therefore $V_d - V_o$. The energy contributed to the region by the tubes emanating from A_d is thus $\frac{1}{2}q_d(V_d - V_o)$. Since A_e possesses a charge q_e, q_e unit tubes emanate from A_e (in addition to the tubes from A_d, which both terminate upon and emanate from A_e, and have already been considered), and all finally reach the boundary of the region at potential V_o. The energy contributed to the region by the q_e unit tubes from A_e is thus $\frac{1}{2}q_e(V_e - V_o)$; and so on for the other conductors in the region.

Consider now a region of lower potential than V_o containing conductors $\cdots$, A_j, A_k, A_l, A_l being the conductor at lowest potential in the region. No tubes emanate from A_l, and the q_l unit tubes terminating upon it all come from, some through, the boundary of the region at potential V_o, though some are discontinuous (induced charges) at the surfaces of A_k and the other conductors, at higher potentials than V_l in the region. The voltage of each of these tubes from the boundary to A_l is thus $V_o - V_l$, and since the total strength of all the tubes is $-q$ (q being negative since V_l is less than V_o), the energy contributed to the region by the tubes of A_l is $\frac{1}{2}q_l(V_l - V_o)$. The energy contributed by the q_k tubes belonging to A_k is likewise $\frac{1}{2}q_k(V_k - V_o)$; and so on for the other conductors of this region, and for other regions.

Summing up these expressions for the whole electric field within A_o, we have for the total energy within A_o

$$W = \tfrac{1}{2}q_1(V_1 - V_o) + \tfrac{1}{2}q_2(V_2 - V_o) + \cdots$$
$$+ \tfrac{1}{2}q_n(V_n - V_o) = \tfrac{1}{2}\Sigma q(V - V_o) \qquad (33)$$

If A_o is connected to the earth, and if we define the potential of the earth as zero potential, (33) becomes

$$W = \tfrac{1}{2}\Sigma q V \qquad (33a)$$

If we suppose A_o removed to infinity, if we suppose the n conductors to be the only electrified bodies in space, and if we define

the potential of A_0, in a region infinitely remote from all electrified bodies, as zero potential, (33*a*) will give the total electrical energy in space surrounding this ideal system.

50–56. A System of Conductors $A_1, A_2, \cdots, A_n$ Surrounded by a Closed Conductor A_0. The voltage from any conductor of the system, such as A_k, to A_0 will be denoted by V with the proper subscript, as V_k, and its charge by q with the same subscript, as q_k.

50. Voltages in Terms of Charges and Charges in Terms of Voltages. The voltage V_k from any one A_k of the system of conductors to A_0 is a linear function of the charges of all the conductors. For, by §§ 34, 35, V_k is proportional to the charge of any one conductor of the system $A_1, A_2, \cdots, A_n$ when all the rest are insulated without charge (except induced charges), and therefore, by § 33, when all the conductors are charged the expression for V_k consists of a series of terms each proportional to the charge of one conductor. Hence

$$\left.\begin{array}{ll} V_1 = p_{11}q_1 + p_{12}q_2 + \cdots + p_{1n}q_n & (1) \\ V_2 = p_{21}q_1 + p_{22}q_2 + \cdots + p_{2n}q_n & (2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ V_n = p_{n1}q_1 + p_{n2}q_2 + \cdots + p_{nn}q_n & (n) \end{array}\right\} \qquad (34)$$

The coefficients of the charges, viz., p_{11}, p_{12}, etc., are called *voltage coefficients*, or *coefficients of potential.* Each has two subscripts, the first identical with that of the conductor in the expression for the voltage from which to A_0 it occurs, the second identical with that of the conductor to whose charge the coefficient belongs, and denotes the ratio of the voltage to A_0 from the conductor with the first subscript to the charge of the conductor with the second subscript when all the conductors except that with the second subscript are insulated without (algebraic) charge. Thus for example,

$$p_{2n} = V_2/q_n$$

when $q_1, q_2, \cdots, q_{n-1}$ are zero in (34) (2).

By solving the n equations (34), each of the charges may be expressed as a linear function of all the voltages. Thus

$$\left.\begin{aligned} q_1 &= s_{11}V_1 + s_{12}V_2 + \cdots + s_{1n}V_n && (1)\\ q_2 &= s_{21}V_1 + s_{22}V_2 + \cdots + s_{2n}V_n && (2)\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ q_n &= s_{n1}V_1 + s_{n2}V_2 + \cdots + s_{nn}V_n && (n) \end{aligned}\right\} \qquad (35)$$

the s's being functions of the p's.

Each s has two subscripts and denotes the ratio of the charge upon the conductor with the first subscript to the voltage to A_0 from the conductor with the second subscript when the voltage to A_0 from each of all the conductors except that with the second subscript is zero (all the conductors except that with the second subscript connected metallically with A_0).

Those coefficients with the two subscripts equal are called *coefficients of capacity*, while those with the subscripts unequal are called *coefficients of induction.*

The p's and s's are functions only of the configuration of the conductors and of the dielectric constant of the medium in which the system is placed.

51. The Coefficients. Any two coefficients of potential with the same subscripts in different order are identical. That is

$$p_{hl} = p_{lh} \qquad (36)$$

To prove this relation, let all the conductors of the system $A_1 \cdots A_n$ except A_h and A_l remain insulated and uncharged (algebraically). The energy of the dielectric within A_0 when A_h and A_l have the charges q_h and q_l will be independent of the manner in which the field is established. The energy obtained by charging A_h first and then A_l may therefore be equated to that obtained by charging A_l first and then A_h. Denoting the energy in the first case by W_1 and that in the second case by W_2, we have

$$
\begin{aligned}
W_1 &= \int_0^{q_k} V_k dq_k [q_l = 0] + \int_0^{q_l} V_l dq_l [q_k = q_k] \\
&= \int_0^{q_k} p_{kk} q_k dq_k + \int_0^{q_l} (p_{lk} q_k + p_{ll} q_l) dq_l = \tfrac{1}{2} p_{kk} q_k^2 \\
&\qquad\qquad + p_{lk} q_k q_l + \tfrac{1}{2} p_{ll} q_l^2 \\
= W_2 &= \int_0^{q_l} V_l dq_l [q_k = 0] + \int_0^{q_k} V_k dq_k [q_l = q_l] \\
&= \int_0^{q_l} p_{ll} q_l dq_l + \int_0^{q_l} (p_{kl} q_l + p_{kk} q_k) dq_k = \tfrac{1}{2} p_{ll} q_l^2 \\
&\qquad\qquad + p_{kl} q_k q_l + \tfrac{1}{2} p_{kk} q_k^2
\end{aligned}
$$

from which (36) immediately follows.

If moreover the expressions in p's for any two coefficients of capacity with the same subscripts in opposite order, , s_{kl} and s_{lk}, are examined, it will be found that they differ ving such coefficients of potential as p_{kl} and p_{lk} interchanged. But $p_{kl} = p_{lk}$, hence

$$s_{kl} = s_{lk} \tag{37}$$

With the aid of (34) and (35) these results may be interpreted as follows:

(36) The voltage to A_0 from any conductor A_k when A_l has a given charge q and all the other conductors of the system $A_1 \cdots A_n$, including A_k, are insulated without algebraic charge, is equal to the voltage from A_l to A_0 when A_k has the same charge q and all the other conductors, including A_l, are without charge.

(37) The charge upon any conductor A_k when the voltage from any other conductor A_l to A_0 is V, and the voltage to A_0 from each of all the other conductors, including A_k, is zero (all the conductors except A_l connected metallically to A_0), is equal to the charge upon A_l when the voltage from A_k to A_0 is V, and the voltage to A_0 from each of all the other conductors, including A_l, is zero (all the conductors except A_k connected to A_0).

The coefficients of voltage are all positive. For if one conductor has a positive charge, *all* the rest being uncharged (algebraically), lines of intensity emanate from it and pass to A_0, some

of them crossing (discontinuously) the other conductors as they go. Thus lines pass from each conductor to A_0. Hence the voltage from any conductor to A_0 is greater than zero, or positive, and the coefficients of voltage are therefore positive ((34) ff.).

The coefficients of capacity are all positive, and the coefficients of induction are all negative. For if any conductor is so charged that the voltage from it to A_0 is positive, while the voltage to A_0 from each of all the rest is zero (the conductors connected to A_0), lines of intensity pass from this conductor to all the others. Hence its charge is positive, and the charges of all the rest are negative. Hence the coefficients of capacity are all positive, and the coefficients of induction are all negative ((35) ff.).

The coefficient of capacity of any conductor A_h is numerically equal to, or greater than, the sum of all the coefficients of induction (with h as first subscript) between A_h and the other conductors. For when A_h is insulated and charged, and the voltage to A_0 from each of all the other conductors (connected to A_0) is zero, the number of unit tubes ending on the other conductors cannot be greater than the number emanating from A_h (since all emanate from A_h), and can equal this number only when A_h is completely surrounded by one or more of the other conductors (of the system $A_1, A_2, \cdots, A_n$) as it is surrounded by A_0 (or when A_0 is removed to infinity and the other conductors are the only two charged bodies in space).

When the dielectric is homogeneous and isotropic throughout, the coefficients of induction and capacity are proportional, and the voltage coefficients inversely proportional, to its permittivity.

Every voltage coefficient with its subscripts equal, as p_{hh}, is diminished, and every coefficient of capacity, as s_{hh}, is increased, by the introduction of another conductor into the field. For V_h, the voltage from A_h to A_0 is equal to $\int E_L dL$ along any path from A_h to A_0; and when another conductor is introduced, whether it is insulated or connected to A_0, tubes of displacement are pushed toward and into this conductor, making the field less

intense, and $\int E_L dL$ less, along some paths not passing through the new conductor (and therefore along all paths). Hence, while q_h remains unaltered, V_h is diminished by the introduction of the new conductor. When A_h is insulated with charge q_h, and all the other conductors are insulated without algebraic charge, $p_{hh} = V_h/q_h$, and is therefore diminished by the introduction of the new conductor. When A_h is insulated with charge q_h and the voltage to A_o from each of all the other conductors (connected to A_o) is zero, $s_{hh} = q_h/V_h$ and is therefore increased by the introduction of the new conductor.

It is easy to see that the effect in question is greater in each case the nearer the new conductor is brought to the conductor whose coefficients are under consideration, and, in general, the greater its volume (space included within its exterior surface, if hollow).

Similar and opposite effects, respectively, are produced by introducing into a part of the field a dielectric of greater or less permittivity than that of the rest of the dielectric within A_o, as will be apparent after reading Chapter IV.

52. Additional Expressions for the Electric Energy. The energy of the electric field within A_o surrounding the system of conductors A_1, A_2, $\cdots$, A_n (§49) can also be expressed as a quadratic function of all the charges $q_1, \cdots, q_n$, or of all the voltages $V_1, \cdots, V_n$. For by (30) and (35)

$$\begin{aligned} W_v = \tfrac{1}{2}\Sigma qV = \tfrac{1}{2}s_{11}V_1^2 + \tfrac{1}{2}s_{22}V_2^2 + \cdots + s_{12}V_1V_2 \\ + s_{13}V_1V_3 + \cdots + s_{23}V_2V_3 + \cdots \end{aligned} \tag{38}$$

and by (30) and (34)

$$\begin{aligned} W_q = \tfrac{1}{2}\Sigma qV = \tfrac{1}{2}p_{11}q_1^2 + \tfrac{1}{2}p_{22}q_2^2 + \cdots + p_{12}q_1q_2 \\ + p_{13}q_1q_3 + \cdots + p_{23}q_2q_3 + \cdots \end{aligned} \tag{39}$$

W_v denoting the energy expressed in terms of the voltages, and W_q the energy expressed in terms of the charges.

By partial differentiation we find that

$$dW_v/dV_1 = s_{11}V_1 + s_{12}V_2 + \cdots + s_{1n}V_n \\ = q_1,\ dW_v/dV_2 = q_2, \text{ etc, and} \tag{40}$$

$$dW_q/dq_1 = p_{11}q_1 + p_{12}q_2 + \cdots + p_{1n}q_n = V_1, dW_q/dq_2 = V_2, \text{ etc.} \tag{41}$$

53. $\Sigma \mathbf{V q}' = \Sigma \mathbf{V}'\mathbf{q}$. Let q_1, V_1, q_2, V_2, $\cdots$, q_n, V_n, and q_1', V_1', q_2', V_2', $\cdots$, q_n', V_n' denote the charges and voltages to A_0 for two static fields surrounding the system of conductors A_1, $\cdots$, A_n within A_0. Then

$$\Sigma Vq' = \Sigma V'q \tag{42}$$

For $\Sigma Vq' = V_1q_1' + V_2q_2' + \cdots + V_nq_n'$. Whence, by (34),

$$V_1q_1' = p_{11}q_1q_1' + p_{12}q_2q_1' + \cdots + p_{1n}q_nq_1'$$

$$V_2q_2' = p_{21}q_1q_2' + p_{22}q_2q_2' + \cdots + p_{2n}q_nq_2'$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$V_nq_n' = p_{n1}q_1q_n' + p_{n2}q_2q_n' + \cdots + p_{nn}q_nq_n'$$

and $$\Sigma Vq' = V_1'q_1 + V_2'q_2 + \cdots + V_n'q_n = \Sigma V'q$$

by adding up in vertical columns the corresponding terms.

54. Change in Energy when Charges and Voltages are Altered. If the charges and voltages to A_0 of the given system of conductors are changed from one set of values q_1, V_1, q_2, V_2, etc., to another set q_1', V_1', q_2', V_2', etc., the increase in the energy of the field is

$$W' - W = \tfrac{1}{2}(\Sigma q'V' - \Sigma qV) \tag{43}$$

This equation may be put in two other forms, sometimes convenient, by (42). Thus

$$\tfrac{1}{2}(\Sigma q'V' - \Sigma qV) = \tfrac{1}{2}(\Sigma q'V' - \Sigma qV' + \Sigma q'V - \Sigma qV) \\ = \tfrac{1}{2}\Sigma(q' - q)(V' + V) \tag{44}$$

$$= \tfrac{1}{2}(\Sigma q'V' + \Sigma qV' - \Sigma q'V - \Sigma qV) \\ = \tfrac{1}{2}\Sigma(q' + q)(V' - V). \tag{45}$$

55. Electric Energy, Mechanical Energy, and Change of Configuration. From §53 it follows that if the system of conductors

$A_1, A_2, \cdots, A_n$ suffers a certain change of configuration, the charges q_1, q_2, etc., remaining constant, and the voltages to A_0 therefore changing from V_1, V_2, etc., to V_1', V_2', etc., the energy, W, *lost* by the electric field *minus* the energy, W', *gained* by the field when the same change of configuration occurs with voltages V_1, V_2, etc., constant, and charges therefore changing from q_1, q_2, etc., to q_1', q_2', etc., is equal to

$$\begin{aligned} W - W' &= \tfrac{1}{2}\Sigma q(V - V') - \tfrac{1}{2}\Sigma V(q' - q) \\ &= \tfrac{1}{2}\Sigma(q' - q)(V' - V) \end{aligned} \tag{46}$$

For in the first case, after the change of configuration, the charges q_1, q_2, etc., correspond to the voltages V_1', V_2', etc.; and in the second case, after the same change of configuration, the charges q_1', q_2', etc., correspond to the voltages V_1, V_2, etc. Hence by (42)

$$\Sigma q V = \Sigma q' V'$$

Substituting $\Sigma q' V'$ for $\Sigma q V$ in the first term of the central member of (46), we obtain $\frac{1}{2}(\Sigma q' V' - \Sigma q V' + \Sigma q V - \Sigma q' V)$, which, on being factored, becomes identical with the last member of (46).

If the energy dissipated in heat during the change of configuration (owing to electric resistance (VIII.)) and that radiated away (both of which are, or may be made, exceedingly small) are neglected, the system gains in the first case an amount of mechanical energy equal to W, the loss of electric energy. If now, after the change of configuration, the voltages to A_0 are brought back to their initial values by means of batteries (or other agents possessing intrinsic e.m.f.s (VIII.)), the state of the system will be the same as after the change of configuration in the second case above, and the electrical energy will surpass the initial energy by the amount W'. The energy W'' supplied to the system by the batteries is equal to the sum of the increases in mechanical and electrical energy, or

$$W'' = W + W'$$

But $$W' = W - \tfrac{1}{2}\Sigma(q'-q)(V'-V)$$

hence $$W'' = 2W - \tfrac{1}{2}\Sigma(q'-q)(V'-V) \tag{47}$$

If the change of configuration is infinitesimal, W, W', and W'' are infinitesimals of the first order, and $\frac{1}{2}\Sigma(q'-q)(V'-V)$ an infinitesimal of the second order. Hence, putting $W = -dW_q$ and $W' = dW_v$, and neglecting $W - W' = -dW_q - dW_v = \frac{1}{2}\Sigma(q'-q)(V'-V)$, we have

$$-dW_q = dW_v \tag{48}$$

and $$W'' = -2dW_q = 2dW_v \tag{49}$$

That is, during any infinitesimal change of configuration the decrease in the electric energy of the system when the charges are kept constant is equal to the increase when the voltages to A_0 are kept constant; and the energy supplied by the batteries, or other sources of electric energy, in the latter case is equal to twice the increase of electric energy — one half going to increase the mechanical energy of the system.

The principle developed in this article will be applied extensively in what follows to find the forcive upon a given conductor A in the field.

Thus suppose the forcive upon A to consist of a force F in the direction OX of increase of the coördinate x of a point of A. Let the configuration of all the other conductors remain fixed while A is displaced in the direction OX a distance dx. Then

$$-dW_q = dW_v = Fdx$$

or $$F = -dW_q/dx = dW_v/dx \tag{50}$$

In the same way, if the forcive consists in a torque T in the direction of an increase of an angle θ,

$$T = -dW_q/d\theta = dW_v/d\theta \tag{51}$$

56. The Discharge by Successive Contacts with A_0 of two Conductors, A_1 and A_2, of the System. Let all the other conductors

within A_0 be kept permanently connected with A_0, thus becoming permanent parts of A_0. Then we have at all times, by (37) and (35),

$$q_1 = s_{11}V_1 + s_{12}V_2$$
$$q_2 = s_{12}V_1 + s_{22}V_2$$

Initially, let A_1 be insulated, with voltage V to A_0 and with charge q, and let A_2 be connected to A_0. In this state we have from the above equations

$$_0q_1 = q = s_{11}V$$
$$_1q_2 = s_{12}V = (s_{12}/s_{11})q$$

the first subscript of q_1 and q_2 denoting the number of times the conductor with the second subscript has been connected to A_0.

Next let A_2 be insulated, and let A_1 be then connected to A_0. After this operation

$$_1q_2 = (s_{12}/s_{11})q$$
$$_1q_1 = s_{12}V_2 = (s_{12}^2/s_{11}s_{22})q.$$

Next let A_1 be insulated and A_2 then connected to A_0. Then

$$_1q_1 = (s_{12}^2/s_{11}s_{22})q$$
$$_2q_2 = (s_{12}^3/s_{12}^2s_{22})q.$$

Then let A_2 be insulated, and let A_1 be connected to A_0. In this state

$$_2q_2 = (s_{12}^3/s_{11}^2s_{22})q$$
$$_2q_1 = (s_{12}^4/s_{11}^2s_{12}^2)q$$

and so on for any number of contacts.

Thus each time either conductor is connected to A_0 its charge is diminished in the ratio $s_{12}^2/s_{11}s_{22}$. After n contacts the charge of A_1 is

$$_nq_1 = (s_{12}^2/s_{11}s_{22})^n q \tag{52}$$

If after the nth contact A_1 is insulated and A_2 connected to A_0, the voltage from A_1 to A_0 is

$$_nV_1 = \frac{1}{s_{11}}(s_{12}^2/s_{11}s_{22})^n q = (s_{12}^2/s_{11}s_{22})^n V \tag{53}$$

When s_{12} is nearly equal to s_{11} and s_{22}, which is the case when the two conductors are parallel and close together, especially when the dielectric constant of the medium between them is larger than that of the rest of the dielectric within A_0, the charge diminishes very slowly with the increase of n.

57. Electric Surface Density and Surface Curvature of Conductors. The electric surface density upon any isolated electrified conductor is, in general, greater at any point of the surface the greater the curvature at the point. For it is obvious that the equipotential surfaces drawn about any such conductor approach more and more nearly the form of spheres about the conductor's "center of charge" as their distances from the conductor increase. That is, at great distances the field is practically radial, and tubes of equal strength have equal cross-sections. If now tubes of equal strength are followed backward toward the conductor, they become narrower; and those which emanate from the more highly curved portions of the surface become narrower more rapidly than those which emanate from less highly curved portions, since the lines bounding each tube emanate normally from the conductor. Thus the area from which a tube of given strength starts is smaller, or the electric surface density upon it greater, the more highly the surface is curved (convex outward) at the point.

In the same manner the density is smaller the more concave the surface.

At a sharp edge or point the density is very great. If the edge or point were really *sharp*, the density there would, of course, be infinite, as tubes would emanate from bases of no dimensions.

Where two parts of a conducting surface make with one another a reëntrant angle the surface density vanishes, since any displacement there would be perpendicular to both parts of the surface, which is impossible.

In addition to the curvature at the given point, the curvature of the neighboring parts of the surface is of importance in deter-

mining the density. Thus the density would be small on a "point" on the inside of a vessel nearly closed, and it might be great in a small cavity in a highly convex portion of the outer surface.

If the conductor is not isolated, the effects here described will be rendered more or less conspicuous according to the signs, magnitudes, and distributions of the charges on neighboring bodies.

In the following chapter many examples illustrating the principle of this article will be found.

58. The Capacity of a Conductor is a very commonly used and convenient, but otherwise objectionable, abbreviation for the permittance of the dielectric (supposed homogeneous and isotropic) enclosed between the conductor and an infinitely remote surrounding conductor when no other conductors or electrified bodies are present.

CHAPTER II.

SIMPLE IDEAL ELECTRIC FIELDS AND CONDENSERS WITH HOMOGENEOUS DIELECTRICS.

In the following articles describing various electric fields all more or less ideal, the dielectric is supposed to be homogeneous and isotropic throughout, and the electrified bodies in each case are supposed to be infinitely remote from all other electrified bodies, unless the contrary is stated. The potential at a point will be taken in this chapter and in Chapter IV. as the line integral of the electric intensity from the point to a region infinitely distant from all electrified bodies.

1. The Spherically Radial Electric Field. Let an electric charge q be concentrated at a point P. The field can be found at once from (1) and (2), Chapter I., or from Gauss's theorem and the principle of symmetry. By symmetry, the electric displacement is directed radially from P (or to P if q is negative) and has the same magnitude at every point of any sphere with center at P. All such spheres are evidently equipotential surfaces. Since the electric flux across any of these equipotentials is q, the flux per unit area across a sphere of radius L, or the electric displacement at a distance L from P, is

$$D = q/4\pi L^2 \tag{1}$$

from which

$$E = q/c4\pi L^2 \tag{2}$$

From (2), the potential at a point distant L from P is

$$V = \int_L^\infty E dL = q/c4\pi \int_L^\infty dL/L^2 = q/c4\pi L \tag{3}$$

the integration being performed along a line of intensity for simplicity.

Maxwell's plane diagram of the field is given in Fig. 14 and described in § 7.

2. The Spherical Condenser. If in the radial field of § 1 infinitely thin conducting sheets are placed coincident with two equipotential spheres of radii L_1 and $L_2 = L_1 + d$, the electric field will remain unaltered, except that it will be rendered discontinuous at the surfaces of the conducting sheet (§ 47, Chapter I.). The charge upon the outer surface of the inner sphere is now q, and that upon the inner surface of the outer sphere is $-q$, the two surfaces with the intervening dielectric forming a condenser whose field is radial and given by (1) and (2).

The charge q at the center of the spheres is the electric image in the inner sphere of the charge on the inner surface of the outer sphere and all external charges; or, if one of the conducting sheets is removed, q is the electric image in the remaining sphere of the complementary charges at an infinite distance (on the infinite sphere at zero potential surrounding q). The conducting substance may be extended into the regions within the inner sphere and without the outer sphere in any manner, or the fields in these regions may be wholly destroyed, without affecting the field of the condenser.

For the voltage between the two conductors (2) gives

$$V_1 - V_2 = \int_{L_1}^{L_2} EdL = q/4\pi c \cdot (1/L_1 - 1/L_2) = qd/4\pi c L_1 L_2 \quad (4)$$

The capacity of the condenser is

$$S = q/(V_1 - V_2) = 4\pi c L_1 L_2/d = 4\pi c L_1^2/d \cdot (1 + d/L_1) \quad (5)$$

and the capacity per unit area of the inner sphere is

$$S' = S/4\pi L_1^2 = c/d \cdot (1 + d/L_1) \quad (6)$$

The energy contained in the dielectric, or the energy of the condenser, is

$$\begin{aligned} W = \tfrac{1}{2}q(V_1 - V_2) = q^2 d/8\pi c L_1^2(1 + d/L_1) = \tfrac{1}{2}S(V_1 - V_2)^2 \\ = 2\pi c L_1^2/d\cdot(1 + d/L_1)\cdot(V_1 - V_2)^2 \end{aligned} \tag{7}$$

Limiting Cases. (1) The parallel plate condenser. If d is kept constant, and L made to increase, the electric field normal to a given portion of the inner or outer sphere obviously approaches uniformity. When L becomes infinite, any finite portion of the condenser becomes a parallel plate condenser (§ 12) of capacity per unit area $S' = c/d$. In any case when d/L is small, the field is approximately uniform (in magnitude) and the capacity per unit area approximately c/d. This field is fully discussed in § 12.

(2) The isolated sphere. If L_2 is made infinite while L remains constant, (4), (5), (6), and (7) become

$$V_1 = q/4\pi c L_1 \tag{8}$$

$$S_1 = q/V_1 = 4\pi c L_1 \tag{9}$$

$$S_1' = c/L_1 \tag{10}$$

and

$$W_1 = \tfrac{1}{2}\,qV_1 = q^2/8\pi c L_1 = \tfrac{1}{2}S_1 V_1^2 = 2\pi c L_1 V_1^2 \tag{11}$$

The coefficients of potential and capacity for the system of two spheres can be easily found from the equations of § 50, I., together with those just developed. Thus

$$\left.\begin{aligned} s_{11} &= q_1/V_1 (\text{when } V_2 = 0) = S \\ p_{11} &= V_1/q_1\,(q_2 = 0) = 1/4\pi c L_1 \\ s_{12} &= q_1/V_2 (\text{when } V_1 = 0) = -S = -s_{11} \\ p_{12} &= V_1/q_2\,(q_1 = 0) = 1/4\pi c L_2 \\ s_{22} &= q_2/V_2 (\text{when } V_1 = 0) = S + S_2 (= 4\pi c L_2) = s_{11} + 1/p_{12} \\ p_{22} &= V_2/q_2\,(q_1 = 0) = 1/4\pi c L_2 = p_{12} \end{aligned}\right\} \tag{12}$$

3. Laplace's Equation for the Spherically Radial Field. As an example of the use of Laplace's equation we will determine $V_1 - V_2$ by a different method. Since the field is radial, the equation may, with the aid of Fig. 12, be put in a much simpler form than that of (21), Chapter I. The simplified form could be obtained from (21), Chapter I., by a mathematical transformation, the proper conditions being put in, but can be developed more simply by starting from first principles.

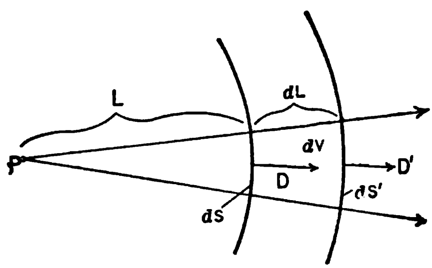


Fig. 12.

The electric flux into the elementary volume $d\tau$ across the surface dS is DdS. The flux out from the volume across dS' is $(D + dD/dL \cdot dL)dS'$, and there is no flux across any other part of the tube. Hence the resultant flux outward is

$$D(dS' - dS) + dD/dL \cdot dLdS = \rho dLdS = \rho d\tau = 0$$

Dividing this equation by dS, writing for dS'/dS its equal $(L + dL)^2/L^2$, and passing to the limit, we obtain, on putting $D = cE$ equal to $-cdV/dL$,

$$2dV/dL + Ld^2V/dL^2 = 0 \qquad (13)$$

which is the form taken by Laplace's equation for a radial field.

From (13) we obtain by integration

$$L^2dV/dL = C_1, \text{ or } dV/dL = C_1/L^2 \qquad (a)$$

where C_1 is a constant to be determined, and

$$V = C_1 \int dL/L^2 \qquad (b)$$

Integrating from $L = L_1$ to $L = L_2$, we have

$$V_2 - V_1 = C_1(1/L_1 - 1/L_2) \qquad (c)$$

Since when $L = L_1$, $dV/dL = -\sigma/c$, (a) gives

$$C_1 = -\sigma L_1^2/c \qquad (d)$$

Hence (c) becomes

$$V_1 - V_2 = \sigma L_1^2/c \cdot (1/L_1 - 1/L_2) = q/4\pi c \cdot (1/L_1 - 1/L_2) \qquad (e)$$

which is identical with (4).

The potential at any point distant L, greater than L_1 and less than L_2 (the field being confined within the space between the spheres of radii L_1 and L_2) from P is

$$V = V_1 + C_1 \int_{L_1}^{L} dL/L^2 \qquad (14)$$

4. The Potential at a Point Due to Any Electric Distribution in a Homogeneous Isotropic Dielectric. For the potential at a point distant $L_1, L_2, \cdots, L_n$ from point charges $q_1, q_2, \cdots, q_n$, respectively, § 33, Chapter I., gives, by means of (3),

$$V = 1/c4\pi \cdot (q_1/L_1 + q_2/L_2 + \cdots + q_n/L_n) = 1/c4\pi\, \Sigma q/L \qquad (15)$$

If the charges, instead of being concentrated at points, which, to be exact, is of course impossible, are distributed over surfaces and through volumes, (15) becomes

$$V = 1/c4\pi \int dq/L = 1/c4\pi \left(\int \sigma dS/L + \int \rho d\tau/L\right) \qquad (16)$$

the first integration extending over all electrified surfaces, and the second throughout all electrified volumes.

While (15) and (16) have been deduced for a space filled up with a single dielectric, they are also true, by § 28, Chapter I., when the field contains any number of conductors. The equations will be extended later to include all cases (IV.).

5. The Law of Inverse Squares. A consideration of equations (1) and (2) shows that the law of inverse squares, which they

state in its simplest form, is due to the continuity of the electric displacement (or the "incompressibility of electricity"), the flux from a charge q being q across every surface surrounding the charge, and to the spherical or three-dimensional nature of space, the flux from a point charge being distributed equally in all directions (when the medium is homogeneous and isotropic, in which case only the law is valid).

6. The Normal Electric Field and the Potential of the Earth. Numerous investigations upon the electrical state of the earth's atmosphere, made at altitudes above its surface ranging from nothing up to 4000 meters, have shown that the atmosphere is the seat of an electric field whose intensity, in normal conditions, is directed toward the earth.

In good weather, the magnitude of this intensity at the earth's surface ranges from about 0.00005 *RES* unit (about 50 volts/meter) to about 0.00040 *RES* unit (about 400 volts/meter), according to season, locality, etc. Thus the electric surface density of the earth's surface, in normal weather, ranges from about −0.00005 *RES* unit to about − 0.00040 *RES* unit. The magnitude of the intensity increases with the altitude above the earth's surface up to heights of some 2000 meters, showing that the atmosphere in this region, like the earth's surface, is negatively charged. In the higher regions of the atmosphere, on the other hand, the intensity decreases with the increase of altitude, without becoming greatly reduced, however, at the greatest altitudes yet investigated. Thus the higher regions of the atmosphere are positively charged; but whether all the tubes of displacement terminating upon the earth and in the lower regions of the atmosphere originate in the upper regions, or whether some of these tubes emanate from other bodies in space, is not yet known. If further investigation demonstrates that at greater altitudes the intensity vanishes, the former alternative will be shown to be correct. The altitudes here considered are so small that no sensible variation in the intensity would occur within them owing to

the increase in the cross-section of the tubes of induction with the distance from the surface of the earth.

The earth itself in any case is negatively charged; and since the electric intensity in the atmosphere is directed toward the earth, its potential is negative if the (wholly unknown) electrification of all other bodies in space beyond the atmosphere is not considered. From the magnitude of the intensity given above and from the great altitude to which the field extends without great diminution in strength, it is obvious that the magnitude of this potential is very great.

It follows from (15) that that part of the potential at the center of a conducting sphere of radius L due to any charges Σq upon its surface is $\Sigma q / 4\pi c L$. Since the sphere is conducting, this expression gives the part of the potential at any point of the sphere due to the surface distribution. From this and § 6, I., it follows that the potential of the earth is not appreciably affected by the the development of any charges retained upon or near its surface. For by § 6, I., Σq is always zero; and L, the distance from the center of the earth, is very great and practically the same for all the charges.

The field surrounding the earth, as a matter of fact, is by no means strictly *static*, and the surface of the earth is never strictly an equipotential.

7. Maxwell's Plane Diagram of the Spherically Radial Field. Maxwell's diagrams are all so drawn that the successive equipotential surfaces differ in potential by the same amount (for example, unity), and that the tubes of induction corresponding to the intervals in the diagram between successive lines of displacement are of equal strength (for example, unit tubes).

(1) *The equipotential surfaces.* The radius of the sphere whose potential is V is

$$L = q / 4\pi c V$$

Hence by giving V in succession the values 1, 2, 3, etc., the radii of the equipotentials with these values of the potential can

be obtained. The circles in which any plane passing through the charge cuts these spheres form the equipotential lines in the diagram. The circles may of course be drawn for any constant increment of potential instead of unity.

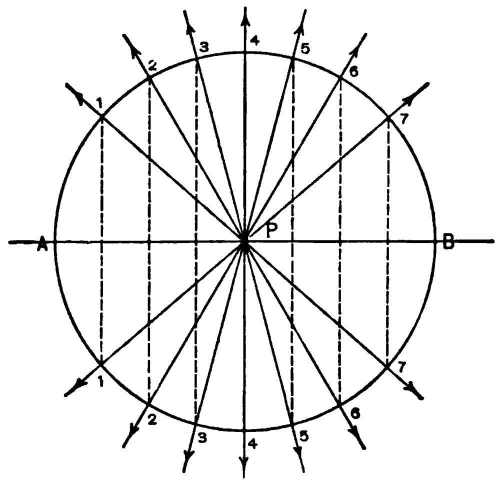


Fig. 13.

(2) *The lines and tubes of displacement.* The lines of displacement are straight lines radiating from the charge. The tubes of displacement in Maxwell's method are formed by rotating the diagram of lines of displacement about a straight line drawn through the charge. We proceed to find the distribution of the lines in the plane diagram when drawn so that the tubes thus formed are of equal strength.

Let a circle of any radius AP, Fig. 13, be drawn about P, the seat of the charge, as center; and let the diameter AB be divided up into q equal parts by straight lines drawn perpendicular to AB and cutting the circle in the points 11, 22, etc. From P let straight lines be drawn through 11, 22, 33, etc. These lines are the lines of Maxwell's diagram.

For if the figure is rotated about AB as axis, the circle traces out a sphere, the lines 11, 22, etc., trace out equidistant parallel

planes which cut the sphere up into zones $1A1$, 2112, etc., of equal area ($1/q$ that of the sphere). Hence the lines $P1$, $P2$, and $P3$, etc., trace out cones $1P1$, $1P2$, $2P3$, etc., through each of which the electric flux is the same and equal to $1/q \times q =$ unity. Hence these cones are the tubes of displacement required, and the lines $P1$, $P2$, $P3$, etc., are the lines of displacement in the diagram.

The field may of course be divided up into tubes of any other strength instead of unity by cutting AB up into the desired number of equal parts in the above construction.

The diagram is given in Fig. 14 for the case in which the strength of each tube is taken as $q/8$.

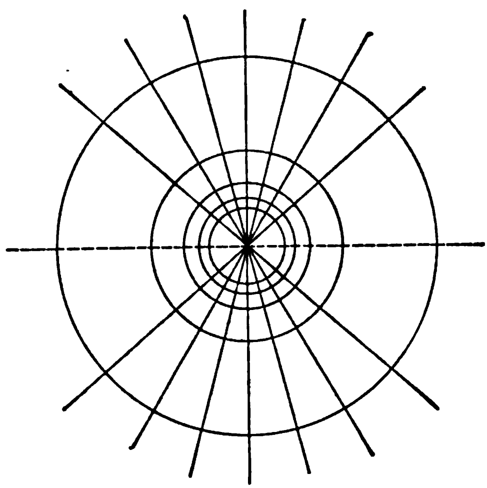


Fig. 14.

8. The Cylindrically Radial Field, or field surrounding a uniformly electrified infinite straight line or circular cylinder. Consider first an electrified straight line, and let the charge on unit length be denoted by q. By symmetry D is everywhere normal to the line and to the circular cylinders about it as axis, which are the equipotential surfaces, and has the same magnitude at every point of any such cylinder. Since the flux across a length A of any equipotential is qA, the flux per unit area across a cylin-

der of radius L, or the electric displacement at distance L from the axis, is

$$D = qA/2\pi LA = q/2\pi L \tag{17}$$

whence

$$E = D/c = q/2\pi cL \tag{18}$$

The potential at a point distant L from the axis is

$$V = q/2\pi c\int_L^\infty dL/L \tag{19}$$

9. The Cylindrical Condenser. If two equipotentials of radii L_1 and $L_2 = L_1 + d$ are replaced by infinitely thin conductors, the electric field will remain unaltered except that it will become discontinuous at the surfaces of the conductors. The charge upon unit length of the outer surface of the inner cylinder is now q, and that upon unit length of the inner surface of the outer cylinder is $-q$, and the two conducting surfaces with the intervening dielectric form a condenser whose field is given by (17) and (18). The charge upon the straight line and that on the inner surface of the outer cylinder together with all the external charges are electric images of one another in the inner cylinder, etc. The conducting substance may be extended into the regions within the inner surface and without the outer surface in any manner, or the fields of these regions may be wholly destroyed, without affecting the field of the condenser.

For the voltage between the plates of the condenser, (18) gives

$$V_1 - V_2 = q/2\pi c\int_{L_1}^{L_2} dL/L = q/2\pi c\cdot\log(1 + d/L_1) \tag{20}$$

The capacity of a length A of the condenser is

$$S = qA/(V_1 - V_2) = 2\pi cA/\log(1 + d/L_1) \tag{21}$$

and the capacity per unit area of the inner cylinder is

$$\begin{aligned} S' &= S/2\pi L_1 A = c/L_1\log(1 + d/L_1) \\ &= c/d(1 - \tfrac{1}{2}d/L_1 + \tfrac{1}{3}d^2/L_1^2 - \cdots) \end{aligned} \tag{22}$$

If d is kept constant and L_1 made to increase, the field normal to a given portion of the inner (or outer) cylinder evidently approaches uniformity; and in the limit, when $L_1 =$ infinity, any finite portion of the condenser becomes a parallel plate condenser (§ 12) of capacity c/d per unit area. In any case when d/L is small the capacity per unit area is approximately c/d.

The energy of a length A of the condenser is

$$\begin{aligned} W = \tfrac{1}{2}qA(V_1 - V_2) &= q^2A/4\pi c \cdot \log(1 + d/L_1) \\ &= \pi Ac/\log(1 + d/L_1) \cdot (V_1 - V_2)^2 \end{aligned} \tag{23}$$

The field of an infinite isolated circular cylinder uniformly charged is given by the above equations on making L_2 infinite and V_2 zero.

$V_1 - V_2$ can be easily obtained by the direct application of the law of inverse squares. Let the field outside the condenser be zero (though the results obtained will be independent of this assumption); then $V_2 = 0$, and V_1 is the potential at any point on or within the inner cylinder, and is therefore the potential at any point P on this axis. Hence, from the figure (Fig. 15),

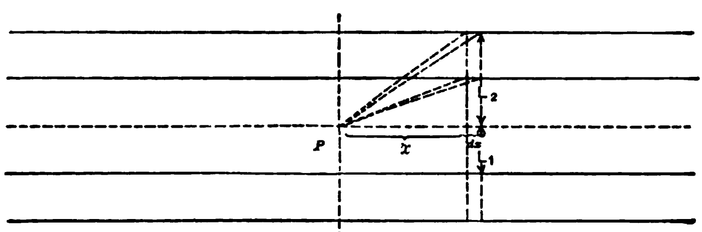


Fig. 15.

$$\begin{aligned} V_1 - V_2 = V_1 = 2\int_0^\infty &[q2\pi L_1 dx/4\pi c(x^2 + L_1^2)^{\frac{1}{2}} \\ &- q2\pi L_2 dx/4\pi c(x^2 + L_2^2)^{\frac{1}{2}}] = q/2\pi c \cdot \log L_2/L_1 \end{aligned} \tag{a}$$

as in (20) above.

10. Laplace's Equation for the Cylindrically Radial Field. $V_1 - V_2$ can also be obtained directly from Laplace's equation.

Without transforming the general equation, we can obtain directly, by a simple process similar to that employed in § 3, the special form it assumes in a cylindrically radial field. Thus we find

$$Ld^2V/dL^2 + dV/dL = 0 \tag{24}$$

Hence, by integration,

$$\left.\begin{aligned} LdV/dL &= C \\ dV &= CdL/L \end{aligned}\right\} \tag{a}$$

or

and

$$V_1 - V_2 = C\int_{L_2}^{L_1} dL/L = q/2\pi c \ \log L_2/L_1 \tag{b}$$

since when $L = L_1$, (a) gives

$$C = -L_1\sigma/c = \cdot \cdot q/2\pi c \tag{c}$$

The potential at any point between the two cylinders, distant L from the axis, is

$$V = V_1 + C\int_{L_1}^{L} dL/L \tag{25}$$

11. Maxwell's Plane Diagram of the Cylindrically Radial Field. This diagram, like that of § 7, is drawn so that the tubes of displacement corresponding to the intervals between the successive lines of displacement are of equal strength, and so that the voltage between successive equipotential lines or surfaces is constant.

Since every line of displacement lies wholly in a plane perpendicular to the axis of the cylinders or electrified straight line, and since the lines of displacement are exactly similar in every such plane, any such plane is chosen as the plane of the diagram, and the tubes of displacement are supposed to be formed by moving the diagram perpendicularly to its plane. We shall suppose the diagram to represent unit depth of the field, all the tubes having this thickness.

1. The equipotential lines in the diagram are circles centered on the axis. Though the potential of every circle, as given by

(19), is infinite, we may draw a system of circles differing in potential successively by a constant finite quantity, as unity, by starting with any equipotential circle of any radius a and potential

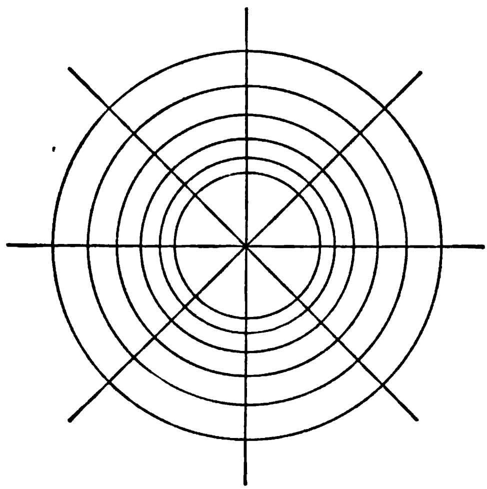


Fig. 16.

V_a (infinite) and applying (20) to find the radius L of the circle whose potential is $V_a - V_L$ less than V_a. Thus we have

$$L = ae^{2\pi c/q \cdot (V_a - V_L)}$$

By giving to $V_a - V_L$ in succession the values 1, 2, 3, etc. (or any set of successive values differing by a constant), as many circles of the system as desired may be obtained.

2. The lines of displacement are straight lines drawn from the center of the circles and dividing each circle into q (or any integral number) of equal parts.

Such a diagram is shown in Fig. 16.

12. The Uniform Electric Field. Let the field be terminated by an infinite plane conducting surface. The surface will be uniformly electrified, the displacement everywhere uniform and normal to the surface, and the equipotentials planes parallel to the surface. The displacement is

$$D = \sigma \tag{26}$$

and the intensity is $$E = D/c = \sigma/c \tag{27}$$

The Parallel Plate Condenser. If an infinitely thin conducting sheet is placed coincident with the equipotential plane distant d from the electrified surface, the electric field will remain unaltered except for the discontinuity introduced at the surfaces of the sheet, and each side of the sheet will therefore have the same electric surface density σ (numerically). The two adjacent surfaces and the dielectric between them form a condenser whose field is uniform and given by (26) and (27), and which will remain unaltered if the conductors are extended into the region outside the condenser.

The voltage between the two conductors is

$$V_1 - V_2 = Ed = \sigma d/c \tag{28}$$

and the capacity of a portion of the condenser of right cross-section A is

$$S = q/(V_1 - V_2) = A/Ed = AD/Ed = Ac/d \tag{29}$$

The capacity per unit area is, as already proved less directly in §§ 2 and 9, $$S' = S/A = c/d \tag{30}$$

The energy of a portion of the condenser of right cross-section A is

$$\begin{aligned} W &= \tfrac{1}{2}q(V_1 - V_2) = \tfrac{1}{2}A\sigma^2 d/c \\ &= \tfrac{1}{2}Ac/d \cdot (V_1 - V_2)^2 = \tfrac{1}{2}EDAd \end{aligned} \tag{31}$$

The force F (positive when tending to increase d, or to separate the plates) upon an area A of either plate, if the tubes in the condenser are the only tubes terminating upon the plates (that is, if there is no external field), is

$$\begin{aligned} F &= -dW/dd\,(\sigma \text{ constant}) = -\tfrac{1}{2}\sigma^2 A/c = -\tfrac{1}{2}cE^2 A = \text{etc.} \\ &= +dW/dd\,[(V_1 - V_2) \text{ constant}] \\ &= -\tfrac{1}{2}Ac(V_1 - V_2)^2/d^2 = -\tfrac{1}{2}cE^2 A = \text{etc.} \end{aligned} \tag{32}$$

by § 55, Chapter I.

This result also follows from § 40, I., or the result there given follows from (32).

$V_1 - V_2$ can be obtained also by the direct application of the law of inverse squares. We assume that there is no field external to the region within the plates, though the results will of course be independent of this assumption, since the internal and external fields are wholly independent of one another. From any point P of the positive plate imagine a straight line drawn perpendicular to both plates. Imagine the surfaces of the conductors divided up into infinitesimal circular zones centered on this line, and let x denote the radius of any zone and dx its width. Then the potential at P, *i. e.*, the potential at all points of the positive plate, is evidently

$$V_1 = \int_0^\infty [-\sigma 2\pi x dx/4\pi c(x^2 + d^2)^{\frac{1}{2}} + \sigma 2\pi x dx/4\pi c x]$$

$$= \sigma/2c \int_0^\infty [1 - x/(x^2 + d^2)^{\frac{1}{2}}] dx = + \sigma d/2c$$

Similarly $$V_2 = -\sigma d/2c$$

Hence $$V_1 - V_2 = \sigma d/c \tag{28}$$

Laplace's Equation for a Uniform Field. The same result can be obtained also from Laplace's equation. In a uniform field, if we take X in the direction of D, the equation (21), Chapter I., simplifies to

$$d^2V/dx^2 = 0 \tag{33}$$

since $D_2 = D_3 = 0$ (or $dV/dy = dV/dz = 0$).

By integration, (33) gives

$$dV/dx = C_1 = -\sigma/c \tag{a}$$

By a second integration

$$V_1 - V_2 = C_1(0 - d) = -C_1 d = \sigma d/c \tag{28}$$

At any point distant x from the positive plate, when x is less than d, we have

$$V = V_1 + C_1 x \tag{34}$$

13. Maxwell's Plane Diagrams of the Uniform Field. (1) The equipotentials are equidistant planes perpendicular to the lines of displacement.

(2) The diagram of the tubes of displacement may be drawn in two different ways. If the tubes are to be formed by moving

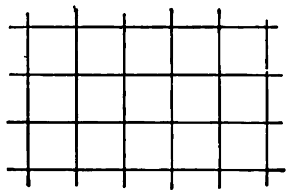


Fig. 17.

the diagram perpendicularly to its own plane, the corresponding lines in the diagram must be drawn equidistant. But if the tubes are to be mapped out by rotating the diagram about a line of displacement as axis, the distances of the successive lines from the axis (or the radii of the outer surfaces of the cylindrical tubes) may be found by giving to the expression $\pi R^2 D$($=$ the flux through a tube of radius R) values which are multiples of the successive whole numbers by a constant, and solving for the corresponding values of R. A diagram of the former kind is given in Fig. 17, and one of the latter kind in Fig. 18.

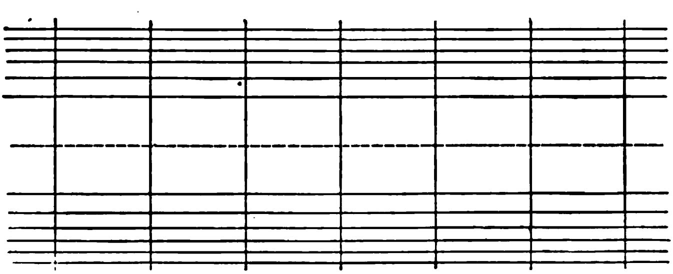


Fig. 18.

14. Maxwell's Plane Diagram of the Resultant of two Fields. If the plane diagrams of two fields are given, both drawn for the same strength of tubes and the same potential differences, and if

both are diagrams which trace out the tubes of displacement by their revolution about the same axis, or by motion at right angles to their plane through the same distance, the resultant diagram of the two fields superposed can easily be drawn.

(1) The equipotentials. Since the potential of each line in both diagrams is known, the potential of every point of intersection when the diagrams are superposed is known. Hence by drawing curves through all the points of intersection which have the same potential we get the resultant equipotential curves. This is equivalent to drawing the curves forming the diagonals of the

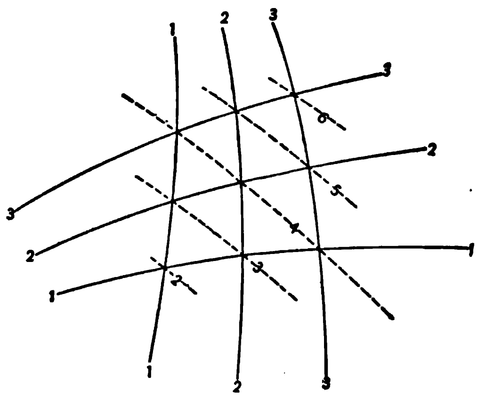


Fig. 19.

quadrilaterals made by the superposition of the two systems of equipotentials, since in passing from one corner to the other the potential of one diagram diminishes as much as that of the other increases. There is no difficulty in choosing the proper diagonal. The difference of potential between the successive curves in the resultant diagram is the same as that between the successive curves in the original diagrams. A particular case is illustrated in Fig. 19, the lines of the resultant diagram being dotted.

(2) The lines of displacement. The lines of displacement in the resultant diagram are the curves forming the diagonals of the quadrilaterals resulting from the superposition of the two

diagrams of lines of displacement. For across every element of such a curve, as *ab* in the figure (Fig. 20), the flux is zero, since the flux through one tube as T in one direction just cancels that through another as T' in the other direction. It is also obvious

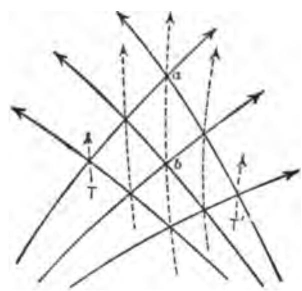


Fig. 20.

from the figure that the flux along any resultant tube is equal to that along any of the original tubes. No difficulty can be experienced in choosing the proper corners of the quadrilaterals to connect.

By compounding diagrams in pairs it is clear that the diagram of the resultant of any number of fields superposed can be obtained by the above method.

15. The Field Terminated by two Equal and Opposite Concentrated Charges. Let the charges, which will be denoted by q and $-q$, be located at A and B, Fig. 21, distant $2d$ apart. The field is evidently symmetrical about the line AB, and is the resultant of two radial fields, § 1. The displacement and intensity at any point P distant L_1 from A and L_2 from B are therefore

$$D = \text{Vector sum of } D_1 (= q/4\pi L_1^2 \text{ directed from } A) \text{ and } D_2 (= q/4\pi L_2^2 \text{ directed toward } B) \tag{35}$$

and

$$E = D/c \tag{36}$$

For the potential at P, (36) gives

$$V = V_1 + V_2 = q/4\pi c \cdot (1/L_1 - 1/L_2) \tag{37}$$

(37) is also the equation of the equipotential surface whose potential is V.

The lines of intensity are the lines orthogonal to the surfaces given by (37). The equation of a line of intensity can be obtained at once by writing down the condition that there is no

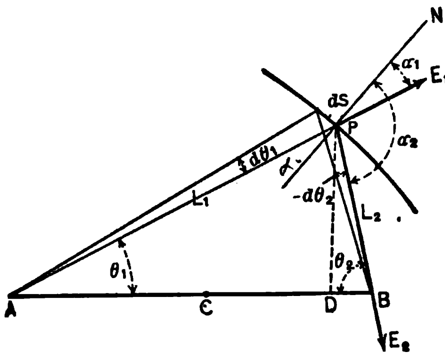


Fig. 21.

component of electric intensity perpendicular to such a line. If ds (Fig. 21) is an element at P of the line of intensity through P, and if a_1 and a_2 are the angles made by E_1 and E_2 with the normal N at P, we have, to express this condition,

$$E_1 \cos a_1 + E_2 \cos a_2 = 0$$

If L_1 and L_2 make with AB the angles θ_1 and θ_2, this condition may be written, as the figure shows,

$$d\theta_1/L_1 + d\theta_2/L_2 = 0$$

Multiplying by PD, the perpendicular to AB from P, and integrating, we have

$$\text{Constant} = C = -\left(\int \sin\theta_1 d\theta_1 + \int \sin\theta_2 d\theta_2\right) = \cos\theta_1 + \cos\theta_2 \quad (38)$$

which is the equation sought, in terms of θ_1 and θ_2.

By giving to C different values, the equation of any line of intensity may be obtained. To find C for the line which cuts

the plane normal to AB through its central point C at a distance x from C, we have, for this point, $\theta_1 = \theta_2$, and

$$L_1 = L_2 = (d^2 + x^2)^{\frac{1}{2}}$$

Therefore

$$C = 2 \cos \theta_1 = 2 \cos \theta_2 = 2d/(d^2 + x^2)^{\frac{1}{2}} \qquad (39)$$

The lines of displacement and the equipotential lines, drawn by the method of Maxwell, § 14, are shown in Fig. 22. (See (Maxwell's *Treatise*, § 123.)

If the distance $2d$ is diminished indefinitely and the charges q and $-q$ increased in such a way that $q \times 2d =$ constant $= M$, the system becomes a *point doublet* of moment M. This doublet and its field are discussed in § 27.

From (37) it follows that the infinite plane perpendicular to AB through its middle point C is at zero potential.

At any point on this plane the resultant intensity and displacement are normal in the direction AB. If P is distant x from C, the displacement at P will be, by (35),

$$D = 2q/4\pi(d^2 + x^2) \cdot d/(d^2 + x^2)^{\frac{1}{2}} = qd/2\pi(d^2 + x^2)^{\frac{3}{2}} \quad (40)$$

If for the infinite plane equipotential surface through C an infinitely thin conducting sheet is substituted, the field on either side will remain unaltered, and the two point charges will become electric images of one another in the sheet.

If the field on the side toward B is destroyed, or if the conductor is extended toward B in any manner, the field on the side toward A will not be affected, and we have the electric field bounded by a concentrated charge q and the (induced) charge upon an infinite plane conducting surface distant d from q and maintained at zero potential. If q is positive the electric surface density is negative at every point of the surface, since D there has the direction AB. The magnitude of $D = \sigma$ is given in (40). The total charge upon the infinite plane is $-q$, since all the tubes from A terminate upon the plane.

Since the field about A was unaltered by the introduction of the conducting sheet and the destruction of the field on the side toward B, the force between the charged body at A and the plane conductor is the same as the force formerly acting between A and B. That is,

$$F = -q^2/16\pi c d^2 \tag{41}$$

Since concentrated charges do not exist, we shall suppose the charges at A and B distributed over extremely small conducting

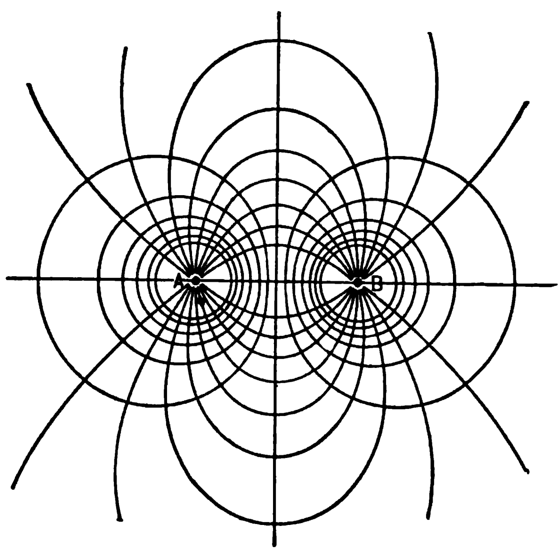


Fig. 22.

spheres each of radius a, so that the field in the region outside the spheres will be practically the same as that already discussed.

From (34), I., the potentials of the spheres A and B are

$$V_1 = p_{11}q + p_{12}(-q) \quad \text{and} \quad V_2 = p_{12}q + p_{22}(-q)$$

the subscripts 1 and 2 being applied to A and B respectively. Since the spheres are very small, these equations become very approximately

$$V_1 = q/4\pi ca - q/4\pi c2d$$

and $$V_2 = -q/4\pi ca + q/4\pi c2d = -V_1$$

from which $$V_1 - V_2 = 2V_1 = q/2\pi c\cdot(1/a - 1/2d) \tag{42}$$

The capacity of the system AB is

$$S = q/(V_1 - V_2) = 2\pi c/(1/a - 1/2d) \tag{43}$$

and the energy of its field is

$$W = q^2/4\pi c\cdot(1/a - 1/2d) = 4\pi c V_1^2/(1/a - 1/2d) \tag{44}$$

When the conducting sheet is placed coincident with the zero equipotential, the capacity of the dielectric between A and this surface is

$$S_1 = q/V_1 = 2S \tag{45}$$

The energy of the dielectric is

$$W_1 = \tfrac{1}{2}W \tag{46}$$

The force tending to increase the distance between the charge at A and the plate can also be found from (44) or (46) by the method of § 55, I. Thus

$$\begin{aligned} F &= -dW_1/dd = -dW/d(2d) = -q^2/16\pi cd^2 \\ &= +dW_1/dd = +dW/d(2d) = -\pi c V_1^2/d^2(1/a - 1/2d)^2 \end{aligned} \tag{47}$$

in agreement with (41), the first differentiations being performed with the charges constant, and the second with the potentials constant.

16. The Electric Field Surrounding Two Concentrated Charges of the Same Sign in the Ratio of 4:1, and its Derivatives. Maxwell's diagram with twenty tubes emanating from one of the charges (A) and five from the other (B) is given in Fig. 23 (from Maxwell's *Treatise*, § 118). One equipotential surface, indicated by the dotted line, consists of two lobes meeting at the point P. At P, which is distant from A two thirds of the distance AB, the intensity vanishes. Within this surface, each charge is surrounded by a separate system of equipotentials, which become

more and more nearly spherical as they become smaller, though no one of them is an exact sphere.

If two of these surfaces, one surrounding each point, are taken to represent the surfaces of two conductors with charges of the same sign in the ratio 4:1, the diagram will represent the equi-

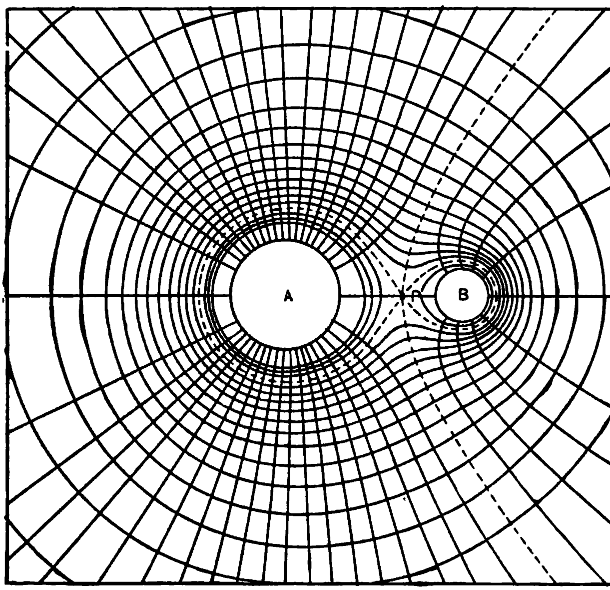


Fig. 23.

potential surfaces and tubes of displacement of the field surrounding the conductors, provided that all the lines within the surfaces are annulled.

The diagram shows that the force between the two bodies will be the same as that between the two points A and B with the same charges. The distribution of the tubes shows that this force tends to pull the bodies apart.

If a conducting surface is placed coincident with the two-lobed equipotential, its electric surface density at P will be zero (cf. § 57, I.).

Outside the two-lobed surface a single system of equipotentials surrounds both charges. By making any of these surfaces

conducting and annulling all the tubes within, we obtain the field surrounding the isolated conductor with a charge upon its surface equal to that of *A* plus that of *B*. The equipotentials surrounding both *A* and *B* approach the form of spheres as their distances from *A* and *B* increase (cf. § 57, I.).

17. The Electric Field Surrounding Two Concentrated Charges of Opposite Signs in the Ratio 4 to 1, and its Derivatives. Maxwell's diagram, with twenty tubes emanating from one charge at *A* and five terminating with the other at *B*, is given in Fig. 24 (from Maxwell's *Treatise*, § 119).

Here again one of the equipotentials, indicated by a dotted line, has two lobes, an inner one surrounding the point *B* and an outer one surrounding both the points *A* and *B*. All the surfaces in the region between the lobes surround *A* only and become more nearly spherical as *A* is approached; while all those in the region within the inner lobe surround *B* only and become more nearly spherical as *B* is approached. The equipotentials lying outside the surface with two lobes become more nearly spherical as their distances from *A* and *B* increase.

One of the surfaces, that with the potential zero, is a sphere, and is indicated by the dotted circle *Q*.

If two of the surfaces, each surrounding one of the two points *A* and *B*, are made conducting, and the fields within them annulled, the diagram gives the tubes and equipotentials surrounding these conductors when charged oppositely in the ratio 4:1.

The diagram indicates that the force between two such charged conductors is one of attraction, and the same as the force between the two charged points *A* and *B*. The field surrounding the charge at *A* or *B* when the sphere *Q* is made conducting and the field on the other side annulled is discussed in § 23.

If we consider points on the axis *AB* beyond the point *B*, we find that the resultant intensity diminishes up to the point *P*, distant from *A* twice the length *AB*, where it vanishes. It then changes

sign and reaches a maximum at M, after which it continually diminishes. The distance of M from A is $\sqrt[3]{4}/(\sqrt[3]{4}-1)\cdot AB = 2.70 \times AB$ (approximately).

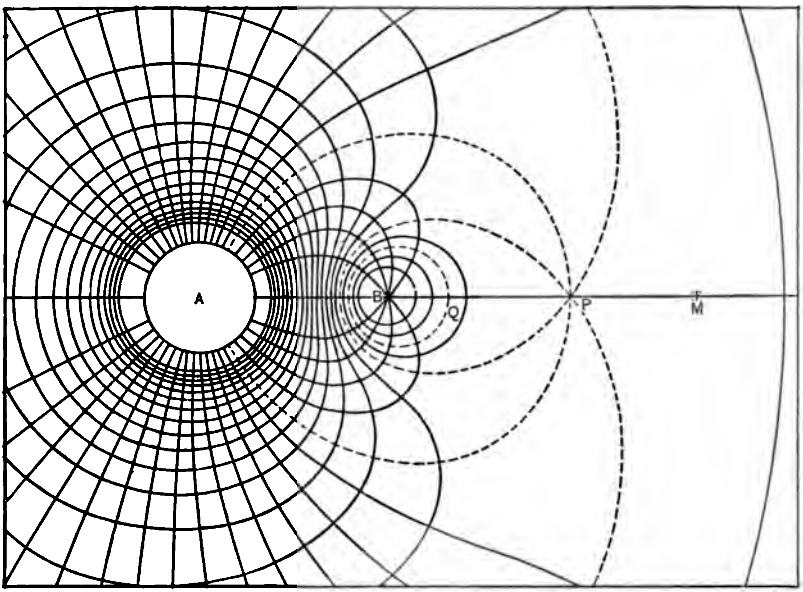


Fig. 24.

18. The Electric Field Surrounding Three Points A, B, and C, with Charges Proportional to 15, — 12, and 20, respectively, so Situated in a Straight Line that AB : BC : AC :: 9 : 16 : 25, and its Derivatives. Maxwell's diagram of the field is given in Fig. 25 (from Maxwell's *Treatise*, § 121).

In this field one of the equipotentials, corresponding to the potential $1/4c$, consists of two spheres intersecting at right angles, with centers A and C, and radii 15 and 20, respectively, as indicated by dotted lines in the diagram. The point B is at the center of the circle of intersection DD, the radius of which is 12, and at all points of which the intensity is zero.

If the sphere A is made conducting and all the lines within it annulled, the diagram will represent the field surrounding the

insulated sphere A with charge 3 upon its surface in the presence of a concentrated charge 20 at C. The part of A within the spherical surface about C will be negatively charged and the rest positively charged, the electric surface density along the circle DD being zero.

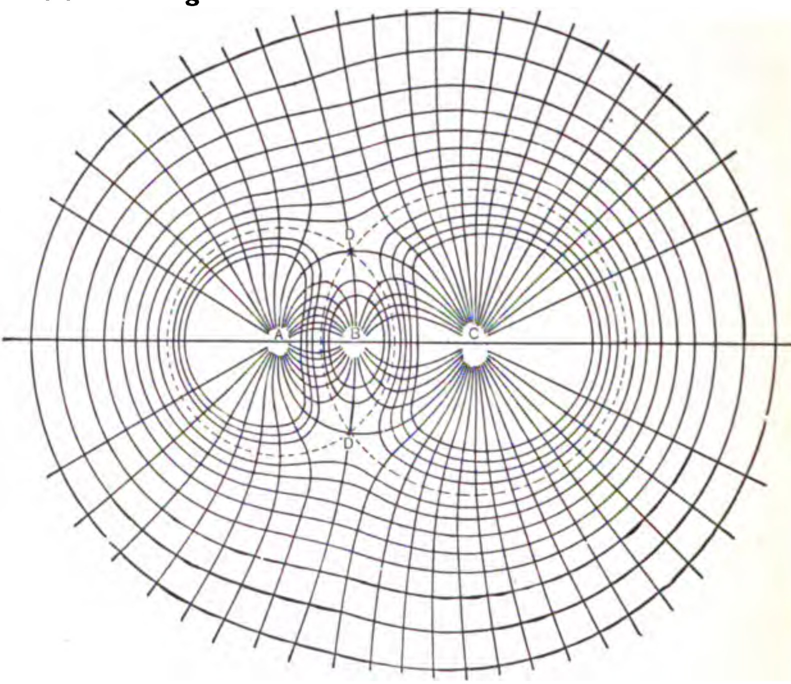


Fig. 25.

In the same way, if C is made conducting, the diagram represents the field surrounding the conducting sphere C insulated with charge 8 upon its surface in the presence of the concentrated charge 15 at A.

These two fields are particular cases of that discussed in § 24.

If both spheres are made conducting, and the lines within annulled, the diagram represents the field surrounding a conducting surface consisting of the external segments of two spheres intersecting at right angles in DD and with charge 23. This is a particular case of the field discussed in § 36.

19. The Electric Field Terminated by Two Infinite Parallel Straight Lines or Circular Conducting Cylinders, with Charges q and $-q$ on Unit Length. Consider first two electrified straight lines, distant $2a$ apart, and cut by a perpendicular plane in the points A_1 and A_2, Fig. 26. By symmetry, the distribution of the lines of displacement is the same in every such plane. Moreover, all the lines emanating from a point A_1 pass to the point A_2 in the plane containing the two points and perpendicular to the two lines.

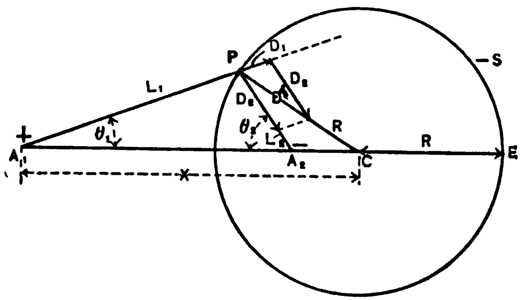


Fig. 26.

The potential at any point P distant L_1 from A_1 and L_2 from A_2 is

$$\begin{aligned} V = V_1 + V_2 &= q/2\pi c\left(\int_{L_1}^{\infty} dL/L - \int_{L_2}^{\infty} dL/L\right) \\ &= q/2\pi c \int_{L_1}^{L_2} dL/L = q/2\pi c \cdot \log L_2/L_1 \end{aligned} \qquad (48)$$

This is also the equation of the section by the plane of the paper of the equipotential surface whose potential is V. By giving to V different values the corresponding surfaces may be obtained.

The displacement at P is

$$\begin{aligned} D = \text{Vector sum of } & D_1 (= q/2\pi L_1 \text{ directed from } A_1) \\ \text{and } & D_2 (= q/2\pi L_2 \text{ directed toward } A_2) \end{aligned} \qquad (49)$$

and the intensity E is D/c.

From (49) the equation of any line of intensity can be obtained by the method of § 15. Proceeding exactly as in that article, we find the equation

$$\theta_1 + \theta_2 = C = \text{constant} \tag{50}$$

which is evidently the equation of the arc of a circle terminating at A_1 and A_2, and cutting perpendicularly the line normal to A_1A_2 at its middle point. If the line whose equation is sought cuts the normal to A_1A_2 at its middle point O at a distance x from O, we have, for this point, $\theta_1 = \theta_2$, and

$$C = 2\theta_1 = 2\theta_2 = 2\theta = 2\cos^{-1}[a/(a^2 + x^2)^{\frac{1}{2}}] \tag{51}$$

The equipotential surfaces given by (48) are circular cylinders, or their lines of intersection with the plane A_1A_2P circles, orthogonal to the lines of intensity. For (48) may be written

$$L_1/L_2 = e^{-2\pi cV/q} = h \tag{52}$$

a constant for the curve, or surface, whose potential is V; and this is the equation of a circle cutting the line A_1A_2 and with its center C on the line A_1A_2 produced.

The radius of the circle whose potential is V is

$$R = 2ha/(h^2 - 1) \tag{53}$$

the distance of its center C from A_1 is

$$A_1C = Rh \tag{54}$$

and the distance of C from A_2 is

$$A_2C = R/h \tag{55}$$

To obtain the resultant displacement D' at P we must obtain the vector sum of D_1 and D_2, Fig. 26, which will be along R normal to the equipotential. Since D_1 and D_2 are directed along L_1 and L_2 respectively, and since, by (49), $D_1/D_2 = L_2/L_1$, the triangle whose sides are D_1, D_2, and D' is similar to the triangle A_1PA_2; so that

$$D'/D_1 = 2a/L_2, \text{ and } D'/D_2 = 2a/L_1$$

whence $$D' = 2aD_1/L_2 = 2aD_2/L_1 = qa/\pi L_1L_2 \tag{56}$$

and the resultant intensity E is equal to D'/c.

The force F upon a length A of either electrified line, considered positive when tending to increase a, is

$$F = -qA\cdot q/c2\pi 2a = -q^2A/4\pi ca \tag{57}$$

The plane diagram of the field, drawn by Maxwell's method, § 14, is given in Fig. 27 (from Webster's *Theory of Electricity and Magnetism*, § 159). The tubes of displacement and the equipotentials are mapped out by moving the diagram perpendicularly to its plane.

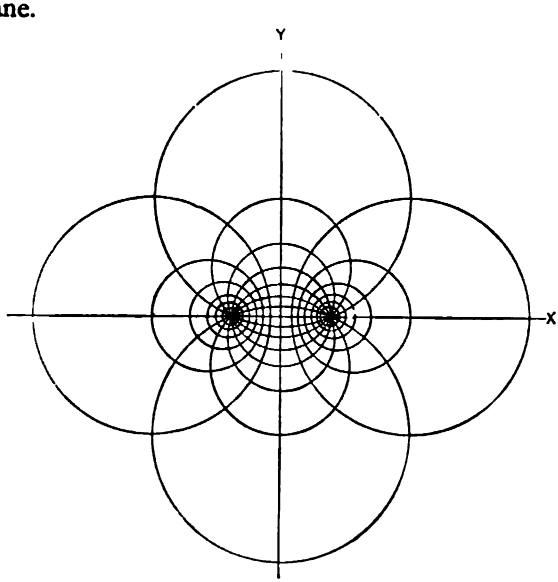


Fig. 27.

If for any equipotential surface the coincident surface of a conductor is substituted, the electric field on the side facing this surface will remain unaltered. The above field therefore includes, as particular cases, the fields bounded by

(1) An infinite straight line and a parallel infinite conducting circular cylinder,

(2) An infinite straight line and a parallel infinite conducting plane,

(3) An infinite conducting circular cylinder and a parallel infinite conducting plane,

(4) Two parallel infinite conducting circular cylinders, internal or external (either or neither surrounding the other), all with charges q and $-q$ upon unit length.

The fields of §§ 8–9 are particular cases of (4) when one of the two lines is removed to infinity.

As systems of practical importance, we shall discuss (4) for the case in which the two cylinders are external to one another, each of the same given radius R, with their axes at a given distance $2d$ apart, and charged to potentials V and $-V$, and (3), which is a particular case of (4).

To obtain the electric field terminated by the two cylinders, we must find the distance a and the charge q upon unit length of the positive cylinder.

From the similar triangles A_1CP and A_2CP (Fig. 26) we have

$$(d+a)(d-a) = R^2 \tag{58}$$

whence

$$a = (d^2 - R^2)^{\frac{1}{2}} \tag{59}$$

From (54) and (55)

$$\begin{aligned} h = (d+a)/R = R/(d-a) &= [d + (d^2 - R^2)^{\frac{1}{2}}]/R \\ &= R/[d - (d^2 - R^2)^{\frac{1}{2}}] \end{aligned} \tag{60}$$

For the cylinder whose potential is $-V$ we have

$$\log h = 2\pi c V/q$$

and therefore

$$\begin{aligned} q = 2\pi c V/\log h &= 2\pi c V/\log[\{d + (d^2 - R^2)^{\frac{1}{2}}\}/R] \\ &= 2\pi c V/\log[R/\{d - (d^2 - R^2)^{\frac{1}{2}}\}] \end{aligned} \tag{61}$$

From (49) and (56) the field can be determined, by making use of (61), at all points.

The capacity of a length A of the system is

$$S = qA/2V = \pi cA/\log h = \pi cA/\log[\{d+(d^2-R^2)^{\frac{1}{2}}\}/R] \quad (62)$$

and the energy in the same length is

$$\begin{aligned} W &= \tfrac{1}{2}qA \cdot 2V = 2\pi cAV^2/\log[\{d+(d^2-R^2)^{\frac{1}{2}}\}/R] \\ &= \tfrac{1}{2}q^2A/\pi c \cdot \log[\{d+(d^2-R^2)^{\frac{1}{2}}\}/R] \end{aligned} \quad (63)$$

If the infinite plane surface of a conductor is placed coincident with the surface of zero potential (the plane passing symmetrically between the conductors) the field on the side facing the conductor will remain unaltered; it is simply half the field just considered.

The capacity of a length A of the condenser formed by the infinite plane and the cylinder with the dielectric is

$$S_1 = qA/V = 2S \quad (64)$$

and the energy is half that contained in the complete field surrounding the two cylinders, or

$$W_1 = \tfrac{1}{2}W \quad (65)$$

The force F acting upon a length of A of either conductor, plane or cylindrical, is given by (57). It can also be obtained by differentiating W_1 with respect to d, or W with respect to $2d$, by the method of § 55, I. Thus

$$\begin{aligned} F &= -q^2A/4\pi ca \\ &= -\pi cAV^2/(d^2-R^2)^{\frac{1}{2}}\log[\{d+(d^2-R^2)^{\frac{1}{2}}\}/R]^2 \end{aligned} \quad (66)$$

20. The Field of a Line Doublet. When $2a$ is small in comparison with L_1 and L_2, we have

$$\begin{aligned} V &= q/2\pi c \cdot \log L_2/L_1 \\ &= \text{(approximately)}\ q/2\pi c \cdot \log(1 + 2a\cos\theta/R) \\ &= q/2\pi c \cdot 2a\cos\theta/R(1 - 2a\cos\theta/2R + \cdots) \end{aligned} \quad (67)$$

if R is written for L_1 and if θ denotes the angle between L_1 and the line A_2A_1.

If now the product $q2a$ is kept constant while a is diminished indefinitely, (67) approaches the limit

$$V = 2aq \cos\theta / 2\pi c R = M \cos\theta / 2\pi c R \qquad (68)$$

where M is written for $2aq$. This system is called a *line doublet*, and M is called the *moment* of the doublet.

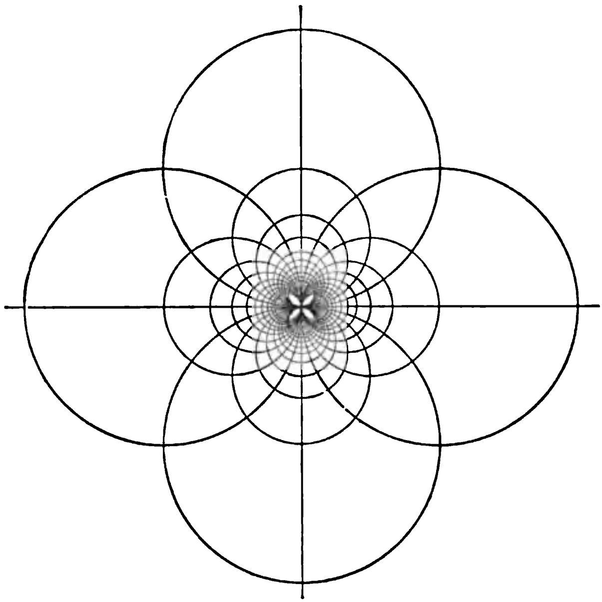


Fig. 28.

The radial and tangential displacements at a distance R from the doublet, at a point where R makes an angle θ with the line A_2A_1, now infinitely short, are

$$D_r = -c dV/dR = M \cos\theta / 2\pi R^2 \qquad (69)$$

and

$$D_t = -cd\, V/dT = -c/R \cdot dV/d\theta = M \sin\theta / 2\pi R^2 \qquad (70)$$

The total displacement is equal to

$$D = (D_r^2 + D_t^2)^{\frac{1}{2}} = M/2\pi R^2 \qquad (71)$$

and makes an angle 2θ with the line A_2A_1 (the axis of the doublet).

The lines of intensity are evidently circles tangent to the axis at O, and the equipotentials circles perpendicular to the axis at

O. The plane diagram of the field is given in Fig. 28 (from Webster's *Theory of Electricity and Magnetism*, § 44), the tubes of displacement and equipotential surfaces being supposed generated by moving the diagram perpendicularly to its plane.

The method of drawing the diagram is easily understood from Fig. 29. Since there is an infinite number of lines of displace-

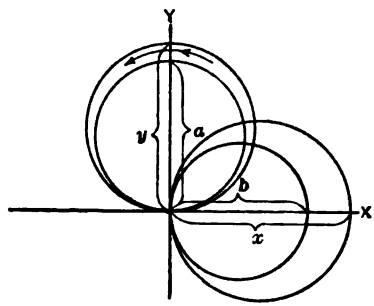


Fig. 29.

ment within a circle of any finite diameter a, only the lines lying outside some such arbitrarily chosen circle can be drawn. The same is true of the equipotential lines.

The flux through the tube between the cylinders of unit depth with diameters a and y is

$$\Pi = \int_a^y M/2\pi \cdot dy/y^2 = M/2\pi \cdot (1/a - 1/y)$$

Hence by giving Π any set of successive values differing by a constant the diameters (y) of the corresponding lines of displacement may be obtained.

The voltage from the circle of equal potential of diameter b to the circle of diameter x is

$$V_b - V_x = M/2\pi c \cdot (1/b - 1/x)$$

Hence by starting with a circle of diameter b and giving $V_b - V_x$ any set of successive values differing by a constant, the diameters (x) of the corresponding equipotential circles may be found.

21. The Electric Field Surrounding an Isolated Conducting Spheroid. First it will be shown that within the region enclosed by a homogeneous material shell whose surfaces are similar and similarly situated ellipsoids there is no gravitational field of force. Such a shell is called an *ellipsoidal homœoid.*

Let a cone, Fig. 30, of infinitesimal angle $d\omega$ at any point A'' in the region cut from the shell the volumes $B''C''$ and $D''E''$. If ρ denotes the density of the shell, g the gravitation constant, L the distance from A'' of any element of volume $d\tau$ of the shell, the intensity at A'' in the direction $A''C''$ due to the masses in $B''C''$ and $D''E''$ is

$$dG = g\int_{A''B''}^{A''C''} \rho d\tau / L^2 - g\int_{A''B''}^{A''E''} \rho d\tau / L^2$$

since the attractions due to the masses in $B''C''$ and $D''E''$ are in opposite directions. Since $d\tau = L^2 d\omega dL$, the integrals reduce to

$$dG = g\rho d\omega (B''C'' - D''E'')$$

and since the plane $N''OC''$ intersects the ellipsoids in two similar and similarly placed ellipses, the same diameter ON'' bisects $B''D''$ and $C''E''$. Hence $B''C'' = C''E''$, and

$$dG = 0$$

In the same manner it may be shown that the intensity at A'' due to the matter within any other infinitesimal cone with vertex at A'' vanishes. Hence

$$G = 0$$

or the region contains no gravitational field.

When a conducting ellipsoid is charged the tubes of displacement are distributed by the tensions and pressures until they touch the surface normally, or until the surface becomes an equipotential. The conducting substance may then be considered replaced by a dielectric of permittivity c equal to that of the exterior medium, for the sake of applying the law of inverse squares, § 28, I. At any point A within this region the intensity and dis-

placement are zero. And therefore, since the law of inverse squares prevails in electrostatics as in gravitation, the law of variation of the electric surface density, or outward displacement at the surface, must be identical with the law of variation of the thickness of the shell in the above gravitational problem. In discussing the electric case the shell must be considered as of infinitesimal thickness, since the charge resides wholly at the surface of the conducting ellipsoid.

The distribution of charge or displacement at the surface of an ellipsoid of revolution, or spheroid, only will be determined here. We proceed to find the distribution of the charge by investigating the law of the variation of the thickness of a spheroidal homœoid.

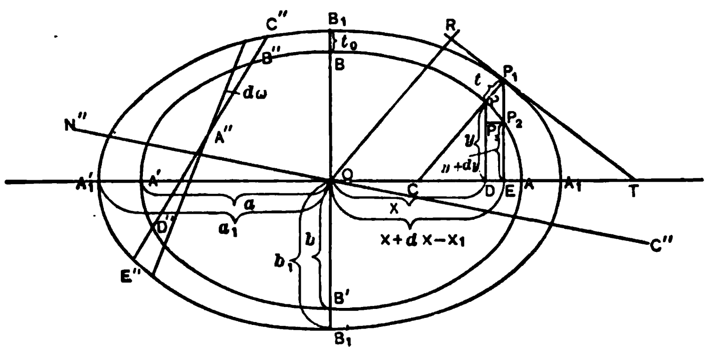


Fig. 30.

Let the given spheroid be generated by the revolution of the ellipse BAB', Fig. 30, about the line BB', and the exterior surface of the infinitely thin homœoid by the revolution of the similar ellipse $B_1A_1B_1'$ about the same axis. We must find the ratio of t, thickness of the shell at any point P, to t_0, the thickness at B. Since the shell is infinitely thin, t is to be measured in the direction of the normal P_1PC to the spheroid at P.

From the similar triangles PP_1P_2 and PCD we have

$$t = P_1P_2 y/PC = ay(y_1 - y)/b(a^2 - e^2x^2)^{\frac{1}{2}} = y(y_1 - y)\,OR/b^2$$

By the equations

$$y_1^2 - y^2 = b_1^2/a_1^2 \cdot (a_1^2 - x_1^2) - b^2/a^2 \cdot (a^2 - x^2),$$

$$b_1/a_1 = b/a \text{ (since the spheroids are similar),}$$

and

$$x = x_1, \quad y = y_1 \quad \text{(in the limit),}$$

we have

$$y(y_1 - y) = b(b_1 - b) = bt_0$$

and the equation for t becomes

$$t/t_0 = b/PC = a/(a^2 - e^2x^2)^{\frac{1}{2}} = OR/b = D/D_0 = \sigma/\sigma_0$$

or

$$D = \sigma = \sigma_0 a/(a^2 - e^2x^2)^{\frac{1}{2}} = D_0 a/(a^2 - x^2e^2)^{\frac{1}{2}} = D_0/b = \sigma_0/b \quad (72)$$

where σ_0 and σ denote the electric surface densities at B and at points of the spheroid distant x from the axis BB', and D_0 and D the corresponding outward displacements.

Thus the surface density increases in passing from B to A, at which point it has the value

$$\sigma = \sigma_0 a/b \quad (73)$$

To obtain the total charge q upon the spheroid, the charge σdS upon the elementary zone cut out by PP_2 as the ellipse revolves about OB must be found and integrated over the whole surface. Thus

$$\sigma dS = \sigma 2\pi x PP_2 = b\sigma_0/PC \cdot 2\pi x dx \cdot PC/y = -2\pi\sigma_0 a^2/b \cdot dy$$

and

$$q = \int \sigma dS = -2\pi a^2\sigma_0/b \int_{+b}^{-b} dy = 4\pi a^2\sigma_0 \quad (74)$$

The potential of the spheroid when the charge is q may be obtained by finding the potential at the center O, since the potential is uniform over and within the spheroid.

The potential at O due to the charge upon the zone dS is

$$dV = \sigma dS/4\pi c(x^2 + y^2)^{\frac{1}{2}} = -\sigma_0 a^2 dy/2bc(x^2 + y^2)^{\frac{1}{2}}$$

$$= -\sigma_0 a^2/2c(a^2 - b^2)^{\frac{1}{2}} \cdot dy/[a^2b^2/(a^2 - b^2) - y^2]^{\frac{1}{2}}$$

and the total potential is

$$V=\int dV=-\sigma_0 a^2/2c(a^2-b^2)^{\frac{1}{2}}\int_{+b}^{-b} dy/[a^2b^2/(a^2-b^2)-y^2]^{\frac{1}{2}}$$
$$=\sigma_0 a^2/c(a^2-b^2)^{\frac{1}{2}}\cdot\sin^{-1}[(a^2-b^2)^{\frac{1}{2}}/a] \quad (75)$$

The *capacity of the isolated spheroid*, or the permittance of the dielectric bounded by the spheroid and an infinitely remote surrounding surface (§ 58, I.), is

$$S=q/V=4\pi c(a^2-b^2)^{\frac{1}{2}}/\sin^{-1}[(a^2-b^2)^{\frac{1}{2}}/a] \quad (76)$$

The isolated sphere. If $b=a$, the equations reduce to those already developed (§ 2) for an isolated sphere.

An infinitely thin circular plate. If $b=0$, the spheroid reduces to an infinitely thin circular conducting plate of radius a, and (72), (74), (75), and (76) become

$$D\ (\text{or}\ \sigma)=aD_0(\text{or}\ \sigma_0)/(a^2-x^2)^{\frac{1}{2}} \quad (77)$$

since $e=[(a^2-b^2)/a^2]^{\frac{1}{2}}=1$,

$$q=4\pi a^2\sigma_0=4\pi a^2 D_0 \quad (78)$$

$$V=\pi a\sigma_0/2c \quad (79)$$

$$S=8ac \quad (80)$$

Near the edges the displacement is very great, D becoming infinite when $x=a$. But the total charge is finite since an edge has no area.

The ratio of the capacity of a thin circular plate to that of a sphere of the same radius is

$$8ac/4\pi ac=2/\pi=1/1.571$$

a relation established experimentally by Cavendish long before the development of the theory.

If two thin circular plates of the same radius a are placed parallel to one another with the distance d between them so small that the field is sensibly uniform between them, and relatively

weak outside, the capacity of the system will be very approximately

$$\pi a^2 c/d$$

The ratio of the capacity of a single isolated plate to this capacity is

$$8d/\pi a$$

The energy of the field connected with two such plates very remote from one another and with charges q and $-q$, respectively, is

$$2(\tfrac{1}{2}q^2/8ac)$$

The energy of the field when the two plates are parallel and separated by the very small distance d is

$$\tfrac{1}{2}q^2 d/\pi a^2 c$$

Hence the work which would be done by the electrical forces in drawing the two plates into the latter configuration from the former is

$$q^2(1/8ac - d/2\pi a^2 c) = q^2/2ac \cdot (1/4 - d/\pi a)$$

It may be shown that the equipotential surfaces surrounding the isolated plates are the confocal spheroids with the edge of the plate as focal line, and that the lines of displacement are the corresponding confocal hyperbolas.

If any one of the spheroidal equipotentials is made conducting, and the lines within annulled, the remainder of the field just described will be the field surrounding this spheroid.

22. The Average Value of the Potential over a Spherical Surface in any electric field whose charges are situated wholly outside of or upon the sphere and whose dielectric is homogeneous and isotropic within the sphere is equal to the potential at its center. To prove this, consider first the spherical surface S of radius R in the radial field from a concentrated charge q distant x from the center of the sphere, q being the only charge in the

field. Let the charge be at A and the center of the sphere at C, Fig. 31.

The area of an elementary zone of the sphere included between two planes perpendicular to AC, at distances y and $y + dy$ from A, is

$$dS = 2\pi R dy$$

The potential of this zone is

$$V_y = q/4\pi c(R^2 - x^2 + 2xy)^{\frac{1}{2}}$$

The integral of the potential over the zone is therefore

$$V_y dS = qRdy/2c\,(R^2 - x^2 + 2xy)^{\frac{1}{2}}$$

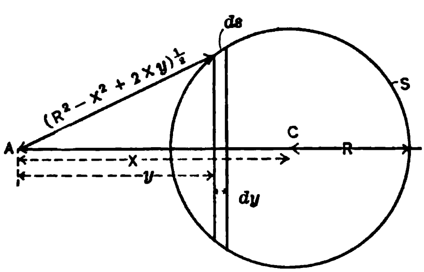


Fig. 31.

To obtain the average value, V, of the potential over the surface of the whole sphere, the integral of this expression must be taken over the sphere, and the result divided by the area of the sphere. Thus

$$V = 1/4\pi R^2 \int V_y dS = q/8\pi c R \int_{x-R}^{x+R} dy/(R^2 - x^2 + 2xy)^{\frac{1}{2}} \tag{81}$$
$$= q/4\pi c x$$

which establishes the proposition for a single concentrated charge.

If, instead of a single concentrated charge, there is any other electric distribution, subject to the limitations above mentioned, we have by the principle of superposition

$$V = 1/4\pi c \cdot \int dq/x$$

which establishes the proposition for a homogeneous dielectric, or such a dielectric and conductors, filling all space. That the proposition is perfectly general will appear from Chapter IV., (17).

As an example, the average potential of the insulated sphere of § 24, which is the same as the potential of any point of the sphere, since it is a conducting surface, is

$$V = q_1/4\pi cx + q/4\pi cR = q_1/4\pi cx$$

since q, the sum of the (induced) charges on the sphere, is zero.

If, in addition to the induced charges, the sphere possesses a charge q, its potential is

$$V = q_1/4\pi cx + q/4\pi cR$$

The same results are obtained by another method in § 24. They also follow immediately from (15) and § 28, I.

23. The Electric Field Surrounding a Concentrated Charge in the Presence of a Spherical Surface at Zero Potential. Consider first the case in which the given charge, q_1, is at A_1 external to the sphere, S, Fig. 26. Let R denote the radius of S and x the distance between its center and the charge q_1.

If a charge q_2 and its position within the surface S can be found such that in the field surrounding q_1 and q_2 S is a surface of zero potential when there are no other charges in the field, then, by § 48, I., the portion of the field outside S will be the only field satisfying the given conditions, and q_1 and q_2 will be the electric images of one another in the surface S if it is made conducting.

If a charge q_2 is placed at A_2, the potential at a point P of the sphere will be

$$V = V_1 + V_2 = 1/4\pi c \cdot (q_1/L_1 + q_2/L_2) = 1/4\pi cL_1 \cdot (q_1 + hq_2)$$

where $h = L_1/L_2$ has the same significance as in § 19.

If q_2 is so chosen that

$$q_1 + q_2 h = 0$$

$V = 0$ for every point upon the sphere. Hence the portion outside S of the field surrounding the charge q_1 at A_1 and q_2 at A_2, where

$$q_2 = -q_1/h \tag{82}$$

is the field required.

A_1 and A_2 are called *inverse points*, or *geometrical images* of one another, in the sphere (§ 30).

The part of the field within S is the field surrounding a concentrated charge q_2 in the interior of a spherical surface at zero potential, the charge being distant $A_2C_2 = x'$ from the center of the sphere.

The resultant displacement, D, at any point of either of the required fields (given charge inside or outside the sphere) is thus the vector sum of D_1 and D_2, the radial displacements which would accompany the charges q_1 and q_2 separately. We shall find the displacement only at the surface of the sphere, to which, an equipotential, it is everywhere normal.

When q_1 is positive, and q_2 therefore negative, D_1, D_2, and D are in the directions A_1P, PA_2, and PC, respectively, their directions being reversed when q_1 and q_2 change signs. Moreover, in magnitude,

$$D_2/D_1 = (q_2/4\pi L_2^2)/(q_1/4\pi L_1^2) = h = x/R = R/x' = L_1/L_2$$

from the geometry of § 19. Therefore the triangle with sides D_1, D_2 and D is similar to the triangle A_1PA_2, and

$$D = 2aD_1/L_2 = 2aD_2/L_1$$

in magnitude.

If D is reckoned positive along the outward normal to S, we have, therefore,

$$\begin{aligned} D &= -2ahq_1/4\pi L_1^3 = -q_1R(h^2-1)/4\pi L_1^3 \\ &= -q_1(x^2-R^2)/4\pi L_1^3R = -q_1/4\pi L_1^3\cdot(x-R)(x+R)/R \end{aligned} \tag{83}$$

which is equal to the electric surface density on the outside of the sphere at P, when the sphere is the surface of a conductor at

zero potential in the presence of the charge q_1 at A_1. The total charge upon the sphere is evidently q_2.

If D is reckoned positive along the inward normal to S, we have

$$D = q_1 R(h^2 - 1)/4\pi L_1^3 = q_1(x^2 - R^2)/4\pi R L_1^3 \tag{84}$$

which is equal to the electric surface density on the inside of the sphere at P when the sphere is the inner surface of a conductor at zero potential surrounding a charge $q_2 = -q_1/h$ at A_2. Since when the internal field is sought, q_2, x', and L_2 are the given quantities instead of q_1, x, and L_1, (84) must be transformed by substituting for q_1, x, and L_1 their equals $-q_2 h = -q_2 R/x'$, R^2/x', and $hL_2 = RL_2/x'$, respectively. On making these substitutions, we have

$$\begin{aligned} D &= -q_2 R(h^2 - 1)/4\pi h^2 L_2^3 = -q_2(R^2 - x'^2)/4\pi R L_2^3 \\ &= -q_2/4\pi L_2^3 \cdot (R - x')(R + x')/R \end{aligned} \tag{85}$$

The total charge on the inner surface of the sphere is $-q_2$. The force between either charged body and the sphere is

$$\begin{aligned} F &= q_1 q_2/4\pi c(2a)^2 \\ &= -q_1^2 Rx/4\pi c(x^2 - R^2)^2 = -q_2^2 Rx'/4\pi c(R^2 - x'^2)^2 \end{aligned} \tag{86}$$

Maxwell's plane diagram of the field, for the case in which $h = -q_1/q_2 = 20/5$ is given in Fig. 24, the sphere, of radius $R = AB/h = \frac{1}{4}AB$, being indicated by the dotted circle Q.

If the radius R, in what precedes, is made to approach infinity, while $(x - R)$ or $(R - x')$ is kept constant, we have, in the limit, a concentrated charge in the presence of an infinite plane surface at zero potential, and the above equations reduce to the equations of § 15.

24. Sphere at Any Potential, or with Any Charge, in the Presence of a Concentrated Charge. The field external to the sphere in § 23 is a particular case of the field surrounding a concentrated charge and a spherical equipotential surface (as that of a

spherical conductor), the potential, V, of the surface, or the outward flux across it (or charge upon it, if a conducting surface), q, being given.

1. If the potential V is given, the required field is found by superposing upon the external field of § 23 the radial field from a charge $q_3 = 4\pi cRV$ at the center of the sphere. For the potential at every point of the sphere due to the fields from q_1 and q_2 is zero, and the potential due to the radial field from q_3 is the same at every point of the sphere and equal to

$$q_3/4\pi cR = 4\pi cRV/4\pi cR = V$$

so that the resultant potential is the same all over the sphere and equal to V. The field outside the sphere is therefore determined (§ 48, I.) and is identical with the field surrounding the charge q_1 and a spherical conductor of radius R and at potential V. The total charge upon the conductor is $q = q_2 + q_3$.

2. If the outward flux q (or total charge, if the surface is that of a conductor) is given, the required field is found by superposing upon the external field of § 23 the radial field from a charge $q_3 = q - q_2$ placed at the center of the sphere. For, as in (1), the surface will remain equipotential, and the flux across it (or the charge upon it) will be $(q - q_2) + q_2 = q$. The position of the equipotential surface being given together with the flux across it and the outside charges, the external field is determined (§ 48, I.) and is identical with the field surrounding a conducting sphere of radius R and charge q. The potential of the sphere is

$$V = q_3/4\pi cR = (q - q_2)/4\pi cR$$

The outward displacement at the surface of the sphere (or the electric surface density if the equipotential sphere is the surface of a conductor) is

$$\left.\begin{aligned} D = D_3 + D'(= D \text{ of preceding article}) &= q_3/4\pi R^2 \\ &\quad - q_1(x^2 - R^2)/4\pi RL_1^3 && (1) \\ = cV/R - q_1(x^2 - R^2)/4\pi RL_1^3 & && (2) \\ = (q + q_1R/x)/4\pi R^2 - q_1(x^2 - R^2)/4\pi RL_1^3 & && (3) \end{aligned}\right\} \quad (87)$$

Maxwell's plane diagram of this field is given for two particular cases in Fig. 25, and is discussed in § 18.

The force between the body with the charge q_1 and the spherical conductor is equal to the force between this body and those with the charges q_2 and q_3. Thus

$$\left.\begin{aligned} F &= q_1 q_2/4\pi c(2a)^2 + q_1 q_3/4\pi c x^2 \\ &= -q_1^2 Rx/4\pi c(x^2 - R^2)^2 + q_1(q + q_1 R/x)/4\pi c x^2 \qquad (1) \\ &= -q_1^2 Rx/4\pi c(x^2 - R^2)^2 + q_1 VR/x^2 \qquad (2) \end{aligned}\right\} \quad (88)$$

If the sphere is insulated without algebraic charge, $q = q_2 + q_3 = 0$, and $q_3 = -q_2 = q_1 R/x$. In this case (87) and (88) become

$$D = q_1/4\pi R \cdot [1/x - (x^2 - R^2)/L_1^3] \qquad (89)$$

and

$$F = -q_1^2 Rx/4\pi c(x^2 - R^2)^2 + q_1^2 R/4\pi c x^2 \qquad (90)$$

If the given charge is internal, the internal field is exactly the same as that in § 23, and the external field is the radial field from the charge q at the center of the sphere.

If a conducting sphere is in the presence of any number of fixed charges, internal and external, the electric images and the electric field can be got at once from what precedes by the principle of superposition.

25. A Conducting Sphere in a Uniform Field. If $q = 0$, § 24, and if q_1 is kept constant and x made to increase without limit, D_1 approaches a uniform direction, parallel to x, and a uniform magnitude, at all points within and near the sphere. But this uniform magnitude is zero. If, however, as x increases, q_1 is made to increase at such a rate that $q_1/4\pi x^2$ remains constant and equal to D_0, then, when $x =$ infinity,

$$D_1 = q_1/4\pi L_1^2 = q_1/4\pi x^2 = D_0$$

and we have an insulated spherical conductor in a field whose displacement is uniform (except for the disturbances due to the presence of the sphere) and equal to D_0.

We can obtain the resultant displacement D at the surface of the sphere by substituting in (89) for q_1 its value $4\pi x^2 D_0$, and for L_1 its value $(x^2 + R^2 + 2Rx \cos\theta)^{\frac{1}{2}}$, where θ is the angle PCE, Fig. 26, reducing, and passing to the limit $x =$ infinity. Thus

$$\begin{aligned}
D &= D_0/R \cdot \{[x(x^2 + R^2 + 2Rx \cos\theta)^{\frac{3}{2}} \\
&\quad - x^2(x^2 - R^2)]/(x^2 + R^2 + 2Rx \cos\theta)^{\frac{3}{2}}\} \\
&= D_0 x[1 + \tfrac{3}{2}(R^2/x^2 + 2R \cos\theta/x) + \cdots \\
&\quad - 1 + R^2/x^2]/R[1 + \tfrac{3}{2}(R^2/x^2 + 2R\cos\theta/x + \cdots)] \qquad (91) \\
&= (3D_0 \cos\theta + \text{terms containing powers of } x \text{ in de-} \\
&\qquad \text{nominators})/(1 + \text{terms containing powers of } x \\
&\qquad \text{in denominators}) \\
&= 3D_0 \cos\theta
\end{aligned}$$

in the limit, the displacement for a given value of the angle θ being thus independent of the radius R of the sphere.

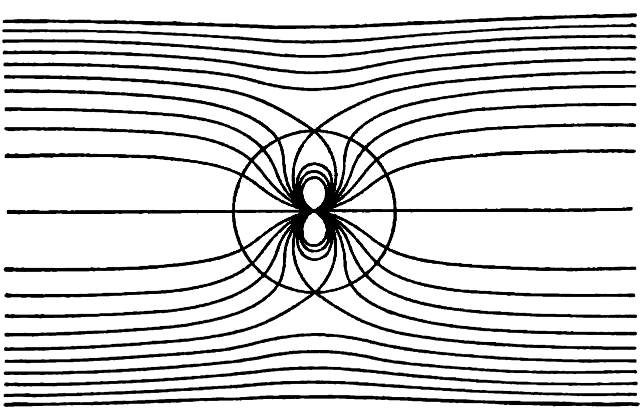


Fig. 32.

At the poles of the sphere, where $\theta = 0°$ and $180°$, $D = +3D_0$ and $-3D_0$, respectively; while its value at the equator, where $\theta = 90°$, is zero.

The displacement at a point of the infinite plane passing through the equator and distant L from the center of the sphere is (§ 26) the sum of D_0 and the displacement $-M \sin 90°/4\pi L^3 = -D_0R^3/L^3$ due to the doublet of moment $M = 4\pi R^3 D_0$ at the

center of the sphere with its axis in the direction of the displacement D_0. Thus at such a point

$$D = D_0(1 - R^3/L^3) \tag{92}$$

The plane diagram of the field drawn by superposing according to the method of Maxwell, § 14, the diagram of the doublet (§ 27) of moment $M = 4\pi R^3 D_0$ and that of the uniform field of displacement D_0 is given in Fig. 32 (from Webster's *Theory of Electricity and Magnetism*, § 194). The diagram of the field surrounding the conducting sphere is obtained from this figure by annulling the lines within the circle (see Fig. 62).

From symmetry, it is evident that there is no resultant force upon the conductor.

A Hemispherical Boss upon an Infinite Plane. All the lines of intensity in the above field (except of course those terminating upon the sphere) cut the infinite plane (an equipotential, with the potential zero) passing through the equator normally. Hence, if this surface is made conducting, the fields on each side will remain unaltered and each will be the field proceeding from (or to) an infinite plane conductor with a hemispherical boss upon it. The surface density upon the boss is given by (91), and that upon the rest of the surface by (92), or by this expression with the opposite sign, according to the half of the original field considered.

26. The Field of an Electrical Point Doublet. Method I. In the problem of § 25, as x approaches the limit infinity, $q_3 = - q_2 = q_1 R/x = 4D_0\pi Rx$ also approaches the limit infinity, while the distance $A_2C = R^2/x$ approaches the limit zero at the same rate. The product $q_3 \cdot A_2C$ therefore remains finite and constantly equal to $4\pi R^3 D_0$. Such a system, consisting of two equal and opposite very great charges at a very small distance apart, when indefinitely near its limit, is called an *electric point doublet*, and the finite product of the positive charge by the distance between them is called the *moment* of the doublet. The straight line directed from the negative to the positive charge is called the *axis* of the doub-

let. In the case considered, the axis of the doublet is the line A_2C, and the moment is

$$M = 4\pi R^3 D_0 \tag{93}$$

The total displacement D in the field of § 25 is therefore the vector sum of the displacement D_d due to the field connected with the doublet and the uniform displacement D_0 directed parallel to the axis of the doublet.

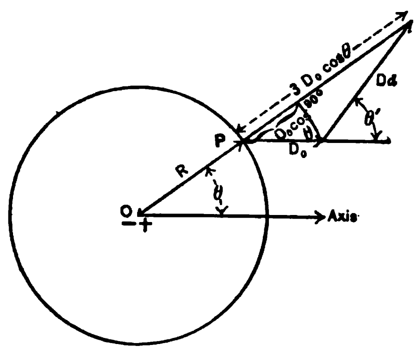


Fig. 33.

In many cases it is necessary to know the field of a doublet alone, that is, to know the displacement D_d. This quantity can be easily found by taking the vector difference of D and D_0, § 25.

From the figure (Fig. 33) it is evident that the radial component of D_d at P, a point whose coördinates are R and θ, is

$$\begin{aligned} D_r = 3D_0 \cos\theta - D_0 \cos\theta = 2D_0 \cos\theta &= 2M\cos\theta/4\pi R^3 \\ &= M\cos\theta/2\pi R^3 \end{aligned} \tag{94}$$

measured in the direction of increase of R.

The tangential component is

$$D_t = D_0 \sin\theta = M\sin\theta/4\pi R^3 \tag{95}$$

measured in the direction of increase of θ.

The total displacement D_d due to the doublet is

$$D_d = (D_r^2 + D_t^2)^{\frac{1}{2}} = M/4\pi R^3 \cdot (3\cos^2\theta + 1) \tag{96}$$

The horizontal and vertical components are

$$D_h = D\cos\theta - D_0 = D_0(3\cos^2\theta - 1)$$
$$= M/4\pi R^3\cdot(3\cos^2\theta - 1) \quad (97)$$

and

$$D_v = D\sin\theta = 3D_0\sin\theta\cos\theta = 3M/4\pi R^3\cdot\sin\theta\cos\theta \quad (98)$$

The angle made by D_d with the axis of the doublet is

$$\theta' = \sin^{-1}D_v/D_d = \sin^{-1}[3\sin\theta\cos\theta/(3\cos^2\theta - 1)] \quad (99)$$

27. The Field of an Electrical Point Doublet. Method II. In § 26 the field of a doublet was obtained by using the results of

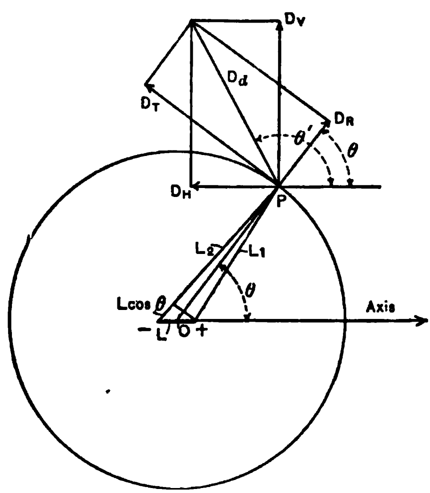


Fig. 34.

§ 25. The field may be found independently as follows. We shall first obtain an approximate solution for the case of two charges q and $-q$ separated by a distance L, short in comparison with $OP = R$, Fig. 34, the product qL being equal to M. Then if M is kept constant while L is increased indefinitely, the system becomes a doublet and the solution, in the limit, exact.

The potential at P is

$$V = q/4\pi c\cdot(L_2 - L_1)/L_1L_2$$

But $L_2 - L_1 = L \cos\theta$ approximately, and $L_1 L_2 = R^2$ approximately, these relations approaching exactness indefinitely as L approaches zero. Hence, in the limit, for a doublet,

$$V = qL\cos\theta / 4\pi c R^2 = M\cos\theta / 4\pi c R^2 \qquad (100)$$

The radial component of the displacement is

$$D_r = cE_r = -cdV/dR = M\cos\theta / 2\pi R^3 \qquad (94)$$

and the tangential component is

$$D_t = cE_t = -cdV/dT = -cdV/d\theta \cdot d\theta/dT = M\sin\theta / 4\pi R^3 \quad (95)$$

since $dT = Rd\theta$, or $d\theta/dT = 1/R$.

From the above equations the horizontal and vertical components, D_h and D_v, and the angle θ' made by D_d with the axis of the doublet, can be readily found. These quantities are given in the preceding article.

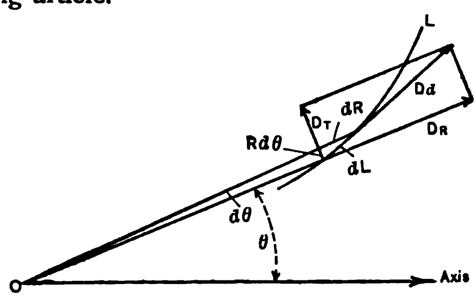


Fig. 35.

The equation of a line of intensity or displacement, L, is easily found from (94) and (95) with the assistance of Fig. 35. For we have evidently

$$D_r / D_t = 2\cos\theta / \sin\theta = dR / Rd\theta$$

from which we obtain

$$dR/R = 2\cos\theta d\theta / \sin\theta$$

The integral of this equation is

$$R = C\sin^2\theta \qquad (101)$$

which is the equation sought, C being a constant for a given line and equal to the distance from the doublet at which the line cuts the plane perpendicular to the doublet's axis through its center.

When C is given for any line, the line can be drawn directly by means of (101), or it can be drawn by the process indicated in Fig. 36. Draw a line OA making an angle θ with OX, which passes through the axis of the doublet, and cutting in the point A a circle with radius C and O as center. Then let fall a perpendicular AB on the line OY, perpendicular to OX through the doublet, and a second perpendicular from B on OA cutting it in P. P is a point on the line required. For $OB = C \sin \theta$, and $OP = OB \sin \theta = C \sin^2 \theta$. See J. Buchanan, *Nature*, Vol. 21, 1880, p. 370.

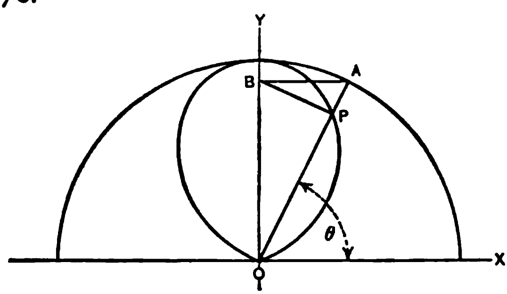


Fig. 36.

If the figure is rotated about the axis OX, the flux through the tube enclosed between the surfaces for which $C = C$ and $C = a$ will be

$$\Pi = \int_a^C M/4\pi C^3 \cdot 2\pi C dC = M/2\pi \int_a^C dC/C^2$$

$$= M/2\pi \cdot (1/a - 1/C)$$

Hence by giving to Π a series of successive values differing by a constant quantity and starting with a curve for which C has an arbitrary value a, the value of C can be determined, and the lines of displacement drawn to correspond to tubes of equal strength, for as much of the field as desired.

The equipotential surfaces corresponding to successive equal potential differences can be drawn by means of (100), which may be written

$$R^2 = M \cos \theta / 4\pi c V$$

and which becomes, for points along OX,

$$R^2 = M/4\pi V c$$

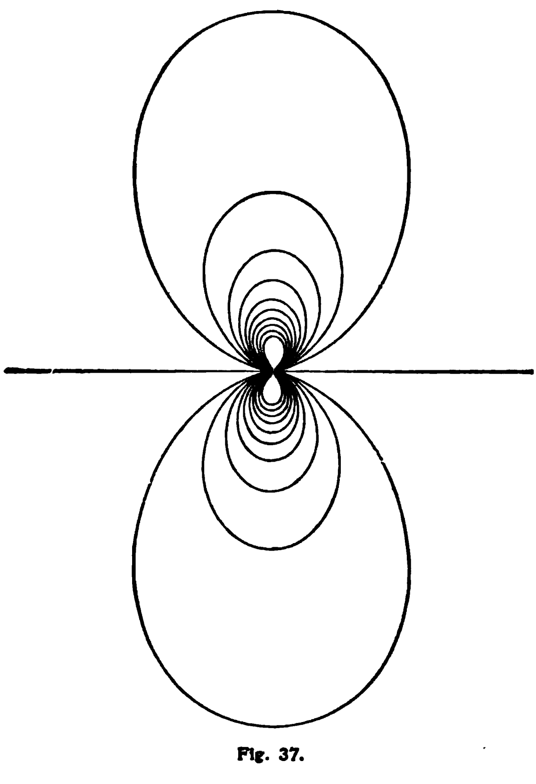


Fig. 37.

The plane diagram of the lines of displacement is given in Fig. 37 from Webster's *Theory of Electricity and Magnetism*, § 122) and the diagram of both lines of displacement and equipotentials for the part of the field lying outside a sphere with the doublet at its center in Fig. 67.

28. The Electric Field Bounded by a Concentrated Charge q and Two Infinite Planes at zero potential meeting at an angle $\theta = \pi/n$, where n is an integer. We shall find the image system of the charge q in the planes. From this system and the given charge q the electric field at all points can then be determined by methods already discussed. Let P denote the position of the given charge, and let its distances from the two planes be denoted by a and b respectively.

Case I. When $n = 1$, $\theta = \pi$, the two planes are coincident, $a = b$, and the image of q is a concentrated charge $-q$ distant from

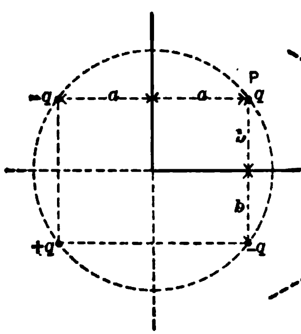


Fig. 38.

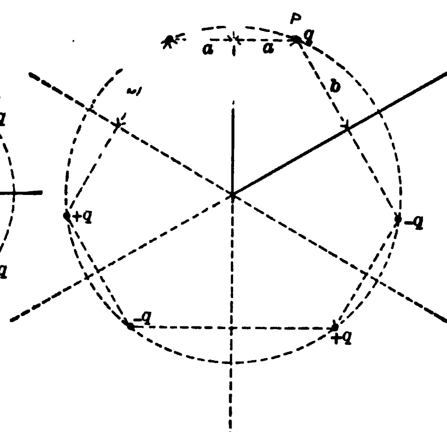


Fig. 39.

q $2a = 2b$ on the other side of the plane on the line through P perpendicular to the plane. This field is fully discussed in § 15.

Case II. When $n = 2$, $\theta = \pi/2$, and the image system consists of the charges $+q$ and $-q$ situated as shown in Fig. 38 at the corners of a rectangle with sides $2a$ and $2b$ and on the circle of radius $(a^2 + b^2)^{\frac{1}{2}}$ with center at the intersection of the planes.

Case III. When $n = 3$, $\theta = \pi/3$, and the image system consists of charges situated as shown in Fig. 39.

Case IV. The images when n is any other integer may be obtained in the same manner. They consist in every case of concentrated charges $+q$ and $-q$ distant a and b from the planes and situated symmetrically on the circle through the given charge at P and with center at the intersection of the planes. The charges, in going around the circle, are alternately + and −.

Case V. When a and b are kept constant and n is made infinite, we have a concentrated charge between two parallel planes,

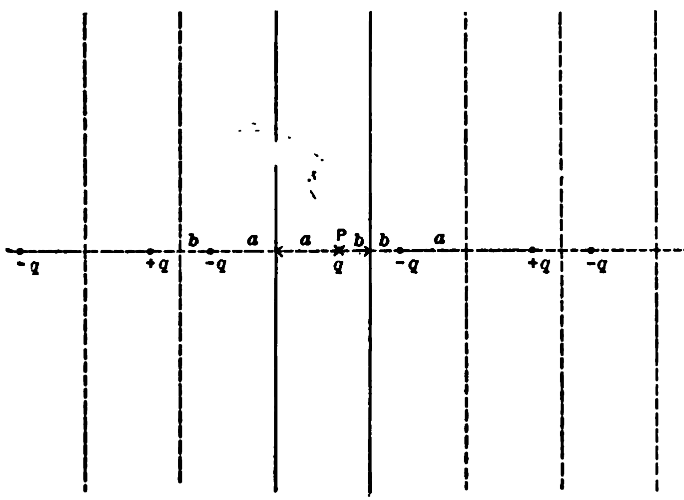


Fig. 40.

Fig. 40. In this case the arc of the circle through P is a straight line, and the images are located along this line at intervals of $2a$ and $2b$.

In all the above fields the total charge upon the two planes is $-q$.

29. The Electric Field Surrounding a Conducting Surface Formed of the External Segments of Two Spheres Intersecting at Right Angles and Maintained at Any Potential V. The field will be found by the method of images.

Let a and b denote the radii of the two spheres with centers at A and C, respectively, Fig. 25, distant d apart. Since the spheres intersect at right angles, we have from the figure

$$BD = ab/(a^2 + b^2)^{\frac{1}{2}} = ab/d$$

and
$$AB\ AC = b^2 = AB\ d$$

$$CB\ CA = a^2 = CB\ d$$

Hence B is the geometrical image of A in the sphere C and the image of C in the sphere A. If therefore a charge $q_1 = 4\pi caV$ is placed at A, a charge $q_3 = 4\pi cbV$ at C, and a charge $q_2 = -4\pi c\ BD = -4\pi c\ ab/d = -q_1\ b/d = -q_3\ a/d$ at B, the (uniform) potential of the sphere A due to the three charges will be $q_1/4\pi ca = V$, since the potential over A due to q_2 and q_3 is zero (§ 23); and the (uniform) potential of the sphere C will be $q_2/4\pi cb = V$, in the same way. Hence both spheres are at potential V in the presence of the three charges. Hence the portion outside the surface of the field connected with these charges is the field sought.

The plane diagram of the field connected with the charges at A, B, and C, when $b/a = 20/15$ and $q_1 = +15$, $q_3 = +20$. $q_2 = -12$, is given in Fig. 25, in which the spheres are shown in dotted lines. At D, the circle of intersection, the displacement is zero, in accordance with § 57, I. The field is further discussed in §18 above.

The charges q_1, q_2, q_3 will be found in another manner in § 36.

The total charge upon the conductor is equal to the algebraic sum of the charges at A, B, and C. Thus

$$q = q_1 + q_2 + q_3 = 4\pi cV[a + b - ab/(a^2 + b^2)^{\frac{1}{2}}] \qquad (102)$$

The capacity of the conductor is

$$S = q/V = 4\pi c[a + b - ab/(a^2 + b^2)^{\frac{1}{2}}] \qquad (103)$$

The charge upon the spherical segment A, or the flux across this segment from the images at A, B, and C, is easily seen to be

$$\begin{aligned} q_1(a + AB)/2a + q_3(b - BC)/2b + \tfrac{1}{2}q_2 = \\ 2\pi cV[a + b + (b^2 - a^2 - ab)/d] \end{aligned} \qquad (104)$$

The charge upon the segment C, found by interchanging a and b in (104) is

$$2\pi c V[a + b + (a^2 - b^2 - ab)/d] \tag{105}$$

If one of the spheres is made infinitely greater than the other, the problem reduces to that of a hemispherical boss upon an infinite plane, § 25.

30. Geometrical Inversion. Let P, Fig. 41, denote any point distant $OP = r$ from a fixed point O, and let a point P' be taken on the line OP such that $OP(=r) \times OP'(=r') = R^2$. Then P and P' are said to be *inverse points* with respect to the sphere, called the *sphere of inversion*, with center O, called the *center of inversion*, and radius R. P and P' are also called the *geometrical images* of one another in the sphere. The process of obtaining P from P', or P' from P, by the relation

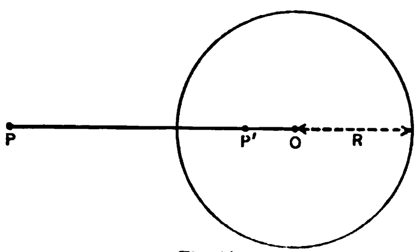


Fig. 41.

$$OP\ OP' = R^2 \text{ or } rr' = R^2 \tag{106}$$

is called *inverting* P or P' with respect to the given sphere.

If every point of a given surface, volume, or curve is inverted with respect to a given sphere, a new surface, volume, or curve will be obtained which is called the *inverse* or *geometrical image* of the given surface, volume, or curve with respect to the given sphere.

31. Inverse of a Sphere. Inverse of a Plane. The image of a sphere (or circle) is another sphere (or circle), the centers of the two spheres (or circles) and of the sphere of inversion being on the same straight line. To prove this, let C, Fig. 42, be the center of the given sphere (or circle) of radius a, distant $OC = b$ from

O, the center of inversion. With O as origin and OB as initial line, the equation of the given circle (or sphere) in polar coördinates (r, θ) is

$$r^2 - 2br\cos\theta + b^2 - a^2 = 0 \tag{107}$$

The equation of the inverse surface is obtained from (107) by substituting for r its equal R^2/r', the coördinates of the inverse being r', θ. Thus we find

$$r'^2 - 2R^2b/(b^2 - a^2)\cdot r'\cos\theta + R^4/(b^2 - a^2) = 0 \tag{108}$$

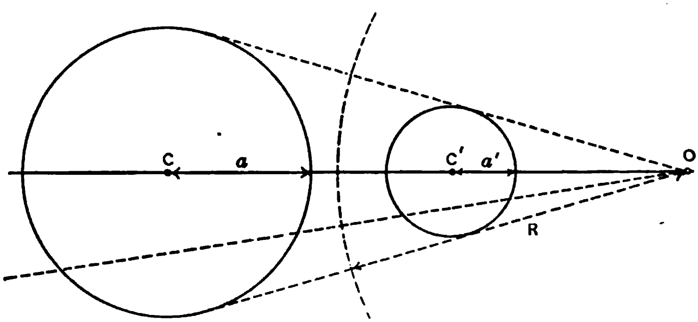


Fig. 42.

which a comparison with (107) shows to be the equation of a sphere (or circle) with center at C' distant

$$b' = R^2b/(b^2 - a^2) \tag{109}$$

from O, and with radius

$$a' = [b'^2 - R^4/(b^2 - a^2)]^{\frac{1}{2}} = R^2a/(b^2 - a^2) \tag{110}$$

When the given sphere passes through the center of inversion, *i. e.*, when $b = a$, Fig. 43, (107) and (108) become

$$r - 2a\cos\theta = 0 \tag{111}$$

and

$$r'\cos\theta - R^2/2a = 0 \tag{112}$$

(112) is the equation of a plane (or straight line) distant

$$p = R^2/2a \tag{113}$$

from O.

Conversely, (112) inverts into (111), so that the image of a plane (or straight line) is a sphere (or circle) passing through the center of inversion and with radius

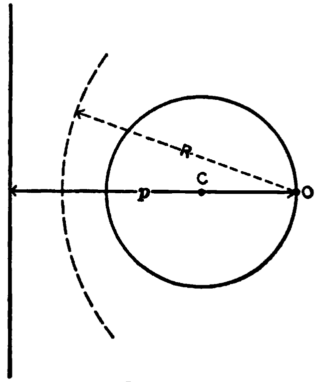


Fig. 43.

$$a = R^2/2p \tag{114}$$

The above propositions can also be easily demonstrated by purely geometrical methods.

32. The Angle at which Two Curves or Surfaces Intersect is Unaltered by Inversion. To establish this proposition for two curves (which will also establish it for two surfaces), let AB and AC,

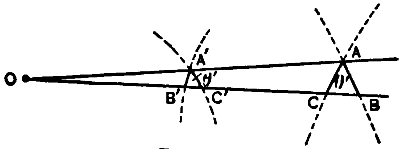


Fig. 44.

Fig. 44, be the elements of two curves intersecting in the point A at an angle θ, and $A'B'$ and $A'C'$ their inverses cutting at the angle θ'. The triangles AOB and $A'OB'$ are evidently similar, since $rr' = R^2$, and likewise the triangles OAC and $OA'C'$. Hence

$$\begin{aligned} \theta' (= \text{angle } OA'C' - \text{angle } OA'B') = \text{angle } OAB \\ - \text{angle } OAC = \theta \end{aligned} \tag{115}$$

33. Electrical Inversion. The electric potential at a point B due to a charge dq at a point A is $dV_b = dq/4\pi cAB$, and the potential at B', the inverse of B, due to a charge dq' at A', the inverse of A, is $dV_{b'} = dq'/4\pi cA'B'$. Thus

$$dV_{b'}/dV_b = dq'/dq \cdot AB/A'B'$$

But, if O is the center of inversion, $AB/A'B' = OA/OB'$. Hence

$$dV_{b'} = dV_b\, dq'/dq \cdot OA/OB' \tag{116}$$

If dq' is so chosen that

$$dq'/dq = -R/OA = -R/r \tag{117}$$

dq' is called the *electric image by inversion* of dq with respect to the sphere of radius R and with center O; and (116) becomes

$$dV_{b'} = -dV_b R/OB' = -dV_b 4\pi cR/4\pi cOB' \tag{118}$$

If now we have any electric distribution whatever, and it produces at B a potential $V_b = \int dV_b$; and if we place at the geometrical images of the charged points charges related to the charges at the original points according to (117), *i. e.*, form a distribution which is the electric image by inversion of the original distribution, the potential at B' will be $V_{b'} = \int dV_{b'}$. Hence (118) gives for the potential at B' due to the electric image by inversion of the original distribution

$$\begin{aligned} V_{b'} = \int dV_{b'} &= -4\pi cR/4\pi cOB' \cdot \int dV_b \\ &= -V_b 4\pi cR/4\pi cOB' \end{aligned} \tag{119}$$

which is the potential which would be produced at B' by a charge $-V_b 4\pi cR$ at O, the center of inversion. If, therefore, we place at O a concentrated charge $q = +V_b 4\pi cR$, the point B' will be at zero potential in the presence of this charge together with the inverted system. If the potential $V_b = V$ is the same for all charged points of the original system, that is, if the original charge is distributed over an equipotential surface, as the surface of a conductor, at potential V, the potential at a point B' of the inverse surface due to the inverse distribution will be

$V_{b'} = -4\pi cRV/4\pi cOB'$. Hence if a charge $q = +4\pi cRV$ is placed at O, the potential at all points, such as B', of the inverse surface will be zero. The introduction of q does not alter the distribution upon the inverse surface, but renders this surface equipotential so that it may be made conducting without disturbing the distribution. The electric field after the introduction of q is the field bounded by a concentrated charge

$$q = 4\pi cRV \tag{120}$$

at O and the inverse of the original surface at zero potential.

Conversely, if we have a charge q concentrated at a point O in the presence of a charged surface at zero potential, we can invert the surface and *its* distribution (*not* including the charge q) with reference to a sphere of radius R and center O and obtain the inverse surface charged to a uniform potential

$$V = q/4\pi cR \tag{121}$$

alone in the field, the charge q being annulled.

Or, if we have a surface at zero potential in the presence of a charge q, concentrated at a point O, and its image system (on the other side the surface), we can invert the image system (not including the charge q) and the given *surface* with respect to a sphere of radius R with center O, and as a result obtain the inverse surface at the uniform potential

$$V = q/4\pi cR$$

and within (or on one side) the inverse image system which produces on the other side the same field as that connected with the surface itself when *charged* to uniform potential V. From the image system the distribution upon the surface, when charged in this manner, or made conducting, can readily be found.

The direct inversion of the distribution on the surface at zero potential would be, in general, a difficult matter. Hence the second of the two processes of inversion just described is usually preferable.

34. The Electric Surface Density upon the Inverse Surface. The electric surface density, σ', at any element dS' of the inverse

surface corresponding to the element dS, with density σ, of the original surface, can easily be found in terms of σ, R, and r', the distance from O to dS'. For we have

$$\sigma' dS'/\sigma dS = dq'/dq = -R/r, \text{ or } \sigma'/\sigma = -R/r \cdot dS/dS'$$

But dS is similar to dS', hence $dS/dS' = r^2/r'^2$ and

$$\sigma'/\sigma = -Rr/r'^2 = -R^3/r'^3$$

Thus
$$\sigma' = -\sigma R^3/r'^3 \tag{122}$$

As stated in § 33, σ' is not altered by the introduction of the charge q at O.

35. The Sphere and Plane. Consider a sphere of radius a uniformly electrified to potential V. Let the sphere be inverted with respect to a sphere of radius R with its center O upon the surface of the given sphere. The sphere inverts into an infinite plane (§ 31) distant $p = R^2/2a$ from O. If now we place at O a charge $q = 4\pi cRV$, the plane will be at zero potential in the presence of the charge q. The electric surface density over the given sphere was uniform and equal to $\sigma = 4\pi caV/4\pi a^2 = cV/a$. Hence the density at a point on the plane distant r' from O is

$$\sigma' = -\sigma R^3/r'^3 = -qR^2/4\pi ar'^3 = -qp/2\pi r'^3 \tag{123}$$

which is the result obtained in § 15, (40), proper attention being paid to sign.

Conversely, we may start with the plane at zero potential electrified to density $\sigma' = -pq/2\pi r'^3$, and by inversion obtain the distribution upon a freely electrified sphere of radius a. Thus the plane inverts into the sphere, and the image of q in the plane, viz., $-q$ distant p from the plane on the other side from q, inverts into the charge $+Rq/2p$ at the center of the sphere; and this brings the sphere to the potential $Rq/2p\pi 4ca = q/4\pi cR = V$.

Next let the center of inversion be taken outside the sphere or inside the sphere, and let the sphere of inversion be so chosen that the given sphere inverts into itself. This makes $R^2 = x^2 - a^2$ when the point O is outside the sphere, and $R^2 = a^2 - x^2$ when

O is inside, if x denotes the distance from O to the center of the sphere. Thus the surface density at a point of the sphere distant r' from O, when a charge $q = 4\pi c R V = 4\pi c (x^2 \sim a^2)^{\frac{1}{2}} V$ is placed at O is

$$\sigma' = -\sigma R^3/r'^3 = -q(x^2 \sim a^2)/4\pi a r'^3 \qquad (124)$$

in accord with (83) and (84).

Conversely, we may pass at once from this distribution to that upon the isolated sphere.

36. Two Spheres Intersecting at Right Angles. The field of § 29 may also be obtained by inverting the distribution upon two infinite planes meeting at right angles and at zero potential under

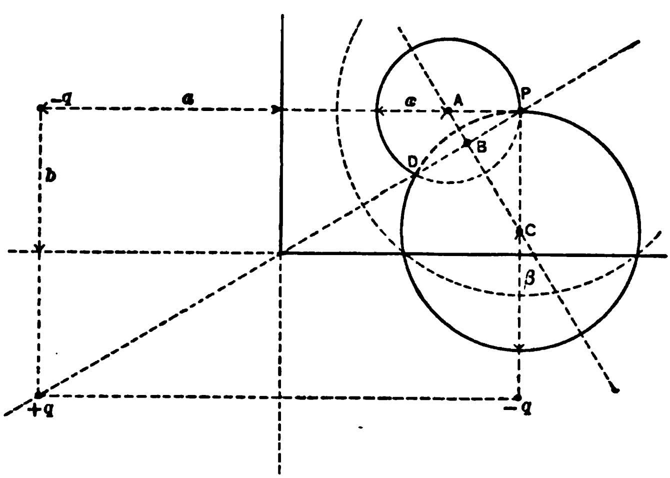


Fig. 45.

the influence of a charge $q = 4\pi c R V$ at a point P distant a and b therefrom (Fig. 45).

Let the planes and the image system in the planes of the charge q at P be inverted with respect to a sphere with center P and radius R. The two planes invert into two spheres intersecting at right angles (corresponding parts of surfaces are shown

in full or dotted lines), and of radii $\alpha = R^2/2a$ and $\beta = R^2/2b$, respectively. The images of q at P invert into charges $qR/2a = q\alpha/R$ at A, $qR/2b = q\beta/R$ at C, and $-qR/2(a^2 + b^2)^{\frac{1}{2}} = -q\alpha\beta/R(\alpha^2 + \beta^2)^{\frac{1}{2}}$ at B. And in the presence of these charges the two spheres are at the potential $V = q/4\pi cR$, and the field outside the spheres is the field required. The total charge upon the spheres, when made conducting, the capacity, etc., may now be found as in § 29.

By inverting the system consisting of two planes meeting at the angle π/n, etc., § 28, the field surrounding two spheres cutting at that angle may be obtained. When $n =$ infinity, this problem merges into that of the next article.

37. Two Spheres in Contact. By inverting the two planes of § 28 with respect to a sphere of radius R and center P we ob-

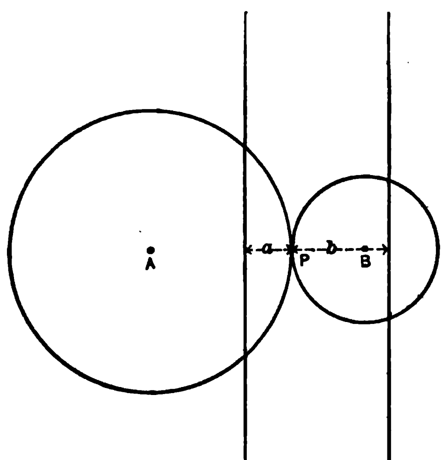


Fig. 46.

tain two spheres of radii $A = R^2/2a$ and $B = R^2/2b$ in contact at P, Fig. 46. All the images to the right of P invert into the region within the sphere B and all to the left of P into the region within the sphere A; and the system of two spheres in contact is brought to the uniform potential $V = q/4\pi cR$, and may be

made conducting, and the inner field destroyed, without affecting the external field.

In the original system the distances from P of the positive charges, $+q$, are

$$2d,\ 4d,\ 6d,\ \cdots \text{ to the right}$$

and

$$2d,\ 4d,\ 6d,\ \cdots \text{ to the left}$$

and the distances from P of the negative charges, $-q$, are

$$2b,\ 2b+2d,\ 2b+4d,\ \cdots \text{ to the right}$$

and

$$2a,\ 2a+2d,\ 2a+4d,\ \cdots \text{ to the left}$$

This system inverts into the system of negative charges

$$-Rq/2d,\ -Rq/4d,\ -Rq/6d,\ \cdots \text{ within the sphere } B$$

and

$$-Rq/2d,\ -Rq/4d,\ -Rq/6d,\ \cdots \text{ within the sphere } A$$

and the system of positive charges

$$Rq/2b,\ Rq/(2b+2d),\ Rq/(2b+4d),\ \cdots \text{ within the sphere } B$$

and

$$Rq/2a,\ Rq/(2a+2d),\ Rq/(2a+4d),\ \cdots \text{ within the sphere } A$$

The total charge of the images within the sphere A, or the total charge upon the surface of A when made conducting, is

$$q_a = Rq/2 \cdot \{[1/a + 1/(a+d) + 1/(a+2d) + \cdots] \\ - (1/d + 1/2d + 1/3d + \cdots)\}$$

and that of the images within B, or upon the surface of the sphere B when a conductor, is

$$q_b = Rq/2 \cdot \{[1/b + 1/(b+d) + 1/(b+2d) + \cdots] \\ - (1/d + 1/2d + 1/3d + \cdots)\}$$

Now

$$[1/a + 1/(a+d) + 1/(a+2d) + \cdots] - (1/d + 1/2d + 1/3d + \cdots) = (1/a - 1/d) + [1/(a+d) - 1/2d] + [1/(a+2d) - 1/3d] + \cdots = \sum_{n=0}^{n=\infty}[1/(a+nd) - 1/(n+1)d] = b/d\sum_{n=0}^{n=\infty} 1/(n+1)(a+nd)$$

$$= 2A^2B/R^2(A+B)\cdot\sum_{n=0}^{n=\infty} 1/[B+n(A+B)(n+1)]$$

Hence

$$\begin{aligned} q_a &= qA^2B/R(A+B)\cdot\sum_{n=0}^{n=\infty} 1/[B+n(A+B)(n+1)] \\ &= 4\pi cVA^2B/(A+B)\cdot\sum_{n=0}^{n=\infty} 1/[B+n(A+B)(n+1)] \end{aligned} \tag{125}$$

Interchanging A and B, we obtain

$$q_b = 4\pi cVB^2A/(B+A)\cdot\sum_{n=0}^{n=\infty} 1/[A+n(B+A)(n+1)] \tag{126}$$

The capacity of the two spherical conductors in contact is

$$S = q/V = (q_a + q_b)/V \tag{127}$$

We shall consider further two particular cases: (1) when $A = B$, (2) when B/A is very small.

(1) $A = B$. In this case

$$\begin{aligned} q_a = q_b &= 2\pi cAV\sum_{n=0}^{n=\infty} 1/(1+2n)(n+1) \\ &= 4\pi cAV(\tfrac{1}{2} + 1/3\cdot 4 + 1/5\cdot 6 + \cdots) \\ &= 4\pi cAV\log 2 = 4\pi cAV \times 0.693 \end{aligned} \tag{128}$$

and

$$S = 2q_a/V = 8\pi cA\log 2 = 4\pi cA \times 1.386 \tag{129}$$

Thus the capacity of the system of two equal spheres in contact is equal to 1.386 times the capacity of a single isolated sphere of the same radius.

The energy of the field surrounding the spheres is

$$W = \tfrac{1}{2}qV = 4\pi cA\log 2\cdot V^2 = q^2/16\pi cA\log 2 \tag{130}$$

From this expression and (11) it is easy to compute the work done against the electrical forces, or the work done upon the electric field, when two equal spheres with equal charges are brought together from an infinite distance (or a great distance, practically) apart.

(2) B/A very small. When $B/A = 0$, the sphere A is alone in the field at potential V; hence $q_a = 4\pi c A V$, a relation which holds approximately when B/A is very small. In this case we have for q_b, approximately,

$$\begin{aligned} q_b &= 4\pi c B^2 V/A \cdot \sum_{n=0}^{n=\infty} 1/(n+1)^2 \\ &= 4\pi c B^2 V/A \cdot (1/1^2 + 1/2^2 + \cdots) = 4\pi c B^2/A \cdot \pi^2/6 \cdot V \quad (131) \\ &= 4\pi c B^2/A \times 1.645\, V \end{aligned}$$

The capacity of the system is, approximately,

$$S = 4\pi c(A + \pi^2/6 \cdot B^2/A) \qquad (132)$$

The electric surface density upon the larger sphere, except near the point of contact, is, approximately,

$$\sigma_a = cV/A \qquad (133)$$

while the average density upon the smaller sphere is, approximately,

$$\sigma_b = cV/A \cdot \pi^2/6 \qquad (134)$$

The relation $\sigma_b/\sigma_a = \pi^2/6$ will hold very approximately when the small sphere B touches any conducting surface A which, like a large sphere, is nearly plane in the neighborhood of the point of contact, the nearly uniform density in that vicinity being, before contact with the small sphere, σ_a.

CHAPTER III.

STANDARD CONDENSERS. CONDENSER SYSTEMS. ELECTROMETERS.

1. **Actual Condensers.** The electric fields and condensers or leydens discussed in Chapter II. are ideal, the conditions assumed being impossible to realise completely in practise. Concentrated charges and infinite conductors do not exist; one or two conductors cannot be infinitely removed from all other conductors; and all the tubes from the first conductor will not, in general, terminate upon the second, unless the second conductor completely surrounds the first or, what amounts to the same thing, is connected with the walls of the room, which thus becomes one, electrically, with the second conductor. The field in this latter case cannot be rigorously computed, however; and though it is possible to construct a condenser of concentric spheres with a high degree of accuracy, or of conductors of other form so arranged that one completely surrounds the other, and such that the electric field can be rigorously computed, yet, to use the system, an insulated wire must pass through an opening in the outer conductor to the inner, and through this opening, however small, some tubes will emerge and pass to the external surface of the outer conductor and to other bodies. The wire and the opening also disturb the field in other ways.

But finite portions of all the fields can be very nearly produced, and the results established above for ideal condensers and fields can be applied without sensible error to actual systems. This can be done, for example, with the systems consisting of two parallel similar conducting surfaces, as two spheres, two planes, or two cylinders, by making the distance between them

small in comparison with their linear dimensions. Then the part of the field between the surfaces — concentric spheres, coaxial cylinders, or parallel planes — becomes practically identical, except near the edges, with those already described, and the part of the field outside this region relatively very weak. The capacity of such a condenser, of any form, is approximately the product of the area of one of its conductors by the permittivity of its dielectric divided by the distance between the conductors, the intensity being practically constant in magnitude throughout the dielectric.

A plane section of the tubes of displacement of a square parallel plate condenser taken parallel to one edge and perpendicular to the plates through their centers is shown in Fig. 47. The

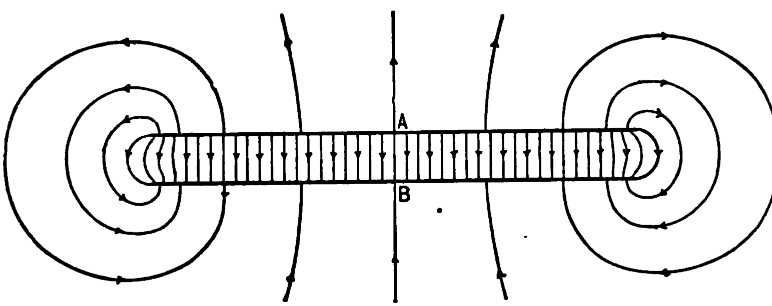


Fig. 47.

tubes in the neighborhood of the section are supposed to be formed by moving the diagram perpendicularly to its plane. The diagram is drawn only approximately. The tubes are closely concentrated and uniformly distributed between the plates, except near the edges, where the field becomes less intense, and sparsely distributed over the outer surface, becoming less numerous as the central points A and B are approached. These results follow from the principle of symmetry and the fact that the voltage $\int EdL$ is the same along every line of intensity from one conductor to the other, which makes the average intensity great where the length of the line is small, and *vice versa*.

As the capacities of ordinary condensers are computed only roughly for construction purposes, and then determined or adjusted accurately, when necessary, by comparison with standard capacities, it is of little importance whether their fields are such that the capacities can be determined accurately by geometrical measurement or not. But this is obviously necessary in the case of condensers designed as absolute standards of capacity. Such standards have been constructed of concentric spheres, coaxial cylinders, and parallel plates. The first form does not need further description here; the last two will be described in the next article.

To eliminate the electric field surrounding the earth, all the apparatus here described will be supposed enclosed within a hollow conductor, such as the walls of a room in a house. The phenomena would not be essentially different, however, outside such an enclosure. The potential of the walls of the enclosure (earthed) will be assumed zero.

2. The Standard Parallel Plate Condenser. The construction of this condenser and its electric field near the center, drawn like the field in § 1, are shown in Figs. 48 and 49. A and A'

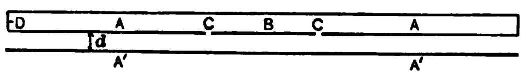


Fig. 48.

are the two plates distant d apart. The central portion, B, of A is separated from the rest by an air gap CC whose breadth is very small in comparison with d. Above and continuous with the plate A is a metal cover D, which forms with A and B a hollow conductor closed except for the gap CC. If B is put in metallic contact with A, and the condenser then charged, the electric field shown in the figures will result, and will remain when B is again insulated from A. Since the region above B is the interior of a hollow conductor practically closed, all the tubes from B will proceed to the upper surface only of A', and

but few tubes will emanate from the upper surface of B and pass through the gap CC to A'. The field below B, being remote from the edges, will be sensibly uniform except in the immediate neighborhood of the gap C, which, however, will not sensibly disturb the uniform field near A'. By symmetry, sensibly half

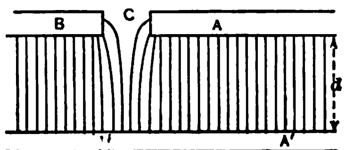


Fig. 49.
(The width of the gap C is greatly exaggerated.)

the tubes which terminate upon the small area of A' just beneath the gap CC must emanate from B and half from A. If b denotes the area of B, and a that of CC, the area of A' receiving tubes from B is thus

$$A = b + \tfrac{1}{2}a \qquad (1)$$

The charge upon B is therefore the same as the charge upon the area $A = b + \frac{1}{2}a$ of A', which is the same charge B would have if its area were $A = b + \frac{1}{2}a$ and there were no gap. The capacity of the condenser formed by B, A', and the tubes connecting them is therefore

$$S = Ac/d = c(b + \tfrac{1}{2}a)/d \qquad (2)$$

The conductor A which surrounds B, and by means of which the field beneath B is made uniform, is called a *guard ring*.

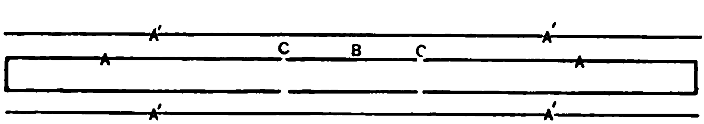


Fig. 50.

The force of attraction between B and A' is the force acting upon the area $A = b + \frac{1}{2}a$ of A'; that is

$$F = \tfrac{1}{2}EDA = \tfrac{1}{2}cE^2A = \tfrac{1}{2}cV^2/d^2 \cdot A \qquad (3)$$

if V denotes the voltage between the plates AB and A'.

The Standard Cylindrical Condenser. The guard ring construction can be applied equally well to the condenser formed of coaxial cylinders. The construction of the condenser is shown in Fig. 50, the inner cylinder serving also as guard ring and protector.

3. Electroscopes and Electrometers. An *electrometer* is an instrument for measuring voltages, or electric potential differences, by means of the forcive acting between electrified bodies. Other instruments for measuring voltages will be described later. An *electroscope* is a crude electrometer, used principally for detecting rather than measuring electrical effects.

4. The Kelvin Absolute Electrometer. This instrument consists of a condenser constructed like the standard parallel plate condenser of § 2 with certain modifications and additions: The plate B (in one of the commonest forms of the instrument) is connected through a small opening in the sheath D with one arm of a gravity balance, so that the force F between B and A' can be determined by weighing. A vertical micrometer screw topped by an insulating support which carries A' enables the distance d to be varied and measured. B is kept in electrical connection with A, and an optical device is provided with the aid of which the planes of A and B are always made coincident (by altering the weights in the balance pans or by varying the distance d) when the force upon B is to be measured.

To measure a voltage, A and A' are first metallically connected, and the weights on the balance pans adjusted until the planes of A and B are coincident. Then the connection between A and A' is broken, and they are brought to the difference of potential V, to be determined. The force F, due to the attraction between the plates is balanced by the addition of known weights, and the distance d is measured. Then by (3)

$$F = \tfrac{1}{2} c V^2 / A d^2$$

whence $$V = d \sqrt{2F/cA} \tag{4}$$

The above method of using the electrometer is called the *idiostatic* method, as the voltage to be determined is the only one employed. Since F is proportional to the square of the voltage, alternating as well as direct voltages can be measured.

In the *heterostatic* method an auxiliary agent with a constant voltage V' is also used. When this voltage alone is applied to the electrometer terminals, we have from the last equation

$$V' = d' \sqrt{2F/cA}$$

if F denotes the force upon B when the distance between the plates is d'.

If now we connect up in series the agent whose voltage V is to be determined and the agent whose voltage is V', both voltages being directed in the same way so that the resultant voltage is $V + V'$, we have, when the voltage $V + V'$ is applied to the electrometer terminals,

$$V + V' = d'' \sqrt{2F/cA}$$

if d'' denotes the distance between the plates when the force F remains the same as before.

Subtracting the first equation from the second gives

$$V = (d'' - d') \sqrt{2F/cA} \tag{5}$$

The advantage which this method has over the other is due to the much greater accuracy with which the micrometer permits the measurement of the difference of the two distances $d'' - d'$ than either separately.

Since F is proportional to V^2, it becomes so small for small voltages that it cannot be accurately measured with this instrument. This form of absolute electrometer is therefore used only for measuring large potential differences, and small voltages are

measured by the quadrant electrometer, the Bichat and Blondlot electrometer, the capillary electrometer of Lippmann, or a form of absolute electrometer recently devised by Perot and Fabry. The first two instruments are described in the succeeding articles.

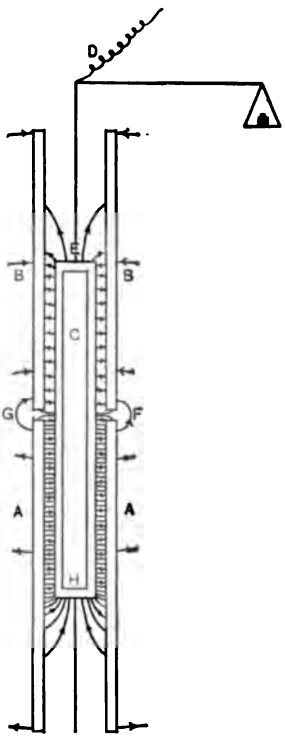


Fig. 51.

Bichat and Blondlot's Electrometer (Modified). This instrument (Fig. 51) consists of a metallic circular cylinder C suspended from the arm of a balance (or connected to another dynamometer) by a fine wire DE, with its axis vertical and coincident with the axis of a longer hollow metallic circular cylinder AB, cut in two at FG and projecting well beyond the ends of C, the difference between the radii (L_1 and L_2, $L_1 > L_2$) of the two cylinders being small in comparison with either radius.

Let A, B and C be charged, the voltage from B to A being denoted by V_{BA}, that from C to A by V_A, and that from C to B by V_B, C being charged by the wire DE, dipping in a conducting liquid. The field, in plane section through the axis, for the case in which $V_A > V_B$ and $V_{BA} = V_A - V_B$ therefore positive, is shown approximately in the figure (all the lines of intensity should touch the conductors normally). Except near and beyond E and H, and G and F, the field is cylindrically radial, and its capacity per unit length is constant and equal to

$$S = 2\pi c \cdot \log L_1/L_2$$

By symmetry, there is no resultant horizontal force acting on C. In general the vertical forces acting on C at H and E are

not equal and opposite. The resultant force can be found as follows: Imagine C to be moved downward an infinitesimal distance dx. The capacity of the condenser AC is increased by Sdx and that of the condenser BC by $-Sdx$, no sensible change occurring in the capacity of the non-cylindrical parts of the field. The increase in the energy of the field, if the voltages are kept constant, is

$$dW = Sdx(V_A^2 - V_B^2)$$

Hence, by § 55, I., the resultant force acting downward upon C is

$$\begin{aligned} F = dW/dx = \tfrac{1}{2}S(V_A^2 - V_B^2) &= \tfrac{1}{2}SV_{BA}(V_B + V_A) \\ &= SV_{BA}(V_B + \tfrac{1}{2}V_{BA}) \end{aligned} \tag{6}$$

(1) If V_B is great in comparison with V_{AB}, the voltage to be measured, this equation becomes, with a negligible error,

$$F = SV_{BA} \cdot V_B \tag{7}$$

Hence, if V_B is kept constant and V_{BA} varied, F is proportional to V_{BA}.

(2) If A and B are connected to the terminal plates of an auxiliary battery consisting of an even number of similar voltaic cells in series, and if one terminal of the cell, condenser, or other agent whose voltage V is to be measured is connected to the central point of this auxiliary battery, the other to the conductor C, we have, if Ψ denotes the e.m.f. of the auxiliary battery, $V_{BA} = \Psi$, $V_B = V - \frac{1}{2}\Psi$, $V_A = V + \frac{1}{2}\Psi$; and (6) becomes

$$F = S\Psi \cdot V \tag{8}$$

so that F is proportional to V if Ψ is kept constant.

(3) If B and C are connected together, $V_B = 0$, $V_A = V_{BA}$, and (6) becomes

$$F = \tfrac{1}{2}SV_{BA}^2 \tag{9}$$

Equations (7), (8), and (9) indicate three methods of comparing voltages with the instrument, the force F being meas-

ured with the balance (or other form of dynamometer). If S is determined from direct measurement, and F measured in dynes, the third method gives an *absolute* determination of the voltage V_{BA}. The first and second methods are called *heterostatic*, as an auxiliary voltage, V_B or Ψ, is employed in addition to that to be measured. The third method is called *idiostatic*, since the voltage to be determined is the only one applied.

5. The Kelvin Quadrant Electrometer. This instrument (Fig. 52) is constructed as follows: A right circular cylindrical me-

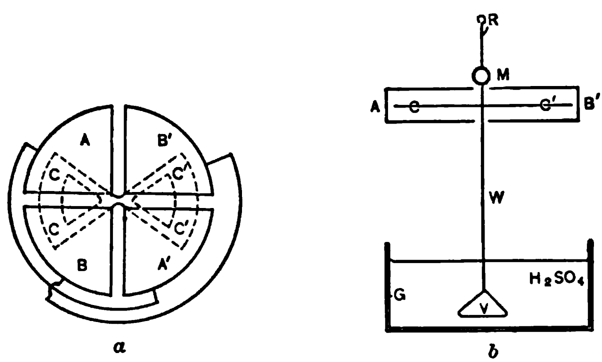


Fig. 52.

tallic box, with its axis vertical, is cut symmetrically into four quadrants A, A', B, B', separately insulated on glass rods, but connected by wires in pairs, A to A' and B to B', so that when the field is static there is never a potential difference between opposite quadrants. A light aluminium needle C, consisting of two equal opposite flat quadrantal arcs CC and $C'C'$ attached by thin radii at their extremities to a central vertical rod R, is suspended from a support by two silk fibers (or other insulating torsion device) in such a way that the arcs CC and $C'C'$ are horizontal, concentric with the quadrant cylinder, and midway between the top and bottom of the box. When the quadrants and the needle are all connected together, so that there is no potential difference between any two parts of the system, the arcs

C, C' are adjusted to lie symmetrically with respect to the two quadrant pairs AA' and BB', as shown in the figure. To the rod R is attached a mirror M, by means of which and a lamp and scale or telescope and scale any deflection, θ, of the needle can be read, and on the other side of the quadrants a vertical platinum wire W, ending in a platinum vane V. The end of the wire and the vane hang free in dry sulphuric acid contained in a glass vessel G, the outer surface of which is partly covered with tin foil. The sulphuric acid serves to make electrical contact with the needle, to dampen the needle's motion, and to form with the tin foil and glass vessel a condenser of considerable capacity, whose function is to keep constant the potential difference between the needle and the case. The whole instrument is enclosed in a tight case, often an extension of the vessel G (whose tin foil covering is then outside) and is kept dry by the sulphuric acid within. The case, largely metal, serves also to screen the needle and quadrants from any external field.

If the instrument is symmetrically made and adjusted, the arcs CC and $C'C'$ form with the two quadrant pairs AA' and BB' two condensers, the capacity of each of which, per unit angle subtended at the center of the system, is the same, let us say S, and constant, except near the edges of the arcs and quadrants, for all but exceedingly large deflections of the needle.

Also, if the instrument is symmetrically made and in adjustment, the needle will obviously not be deflected, even when charged, as long as the quadrants are all connected together. If the needle and the quadrant pairs AA' and BB' are charged, the needle will, in general, be deflected, coming to rest when the angle of deflection, θ, is such that the torque T upon it due to the electrical stresses is balanced by the return torque due to the twist of the suspension. To find the relation between the deflection and the voltage, we may proceed as follows, using the method of § 55, I.

Let V_{AB} denote the voltage from the quadrants AA' to the quadrants BB', V_A the voltage from the needle to AA', and V_B the voltage from the needle to BB'.

When θ is increased by an amount $d\theta$, the capacity of the condenser formed by CC' with AA' is increased by $Sd\theta$, and that of the condenser formed by CC' with BB' is decreased by the same amount. The increase in the energy of the two condensers is then

$$\begin{aligned} dW &= \tfrac{1}{2}Sd\theta\, V_B^2 - \tfrac{1}{2}Sd\theta\, V_A^2 = \tfrac{1}{2}Sd\theta(V_B - V_A)(V_B + V_A) \\ &= \tfrac{1}{2}Sd\theta\, V_{AB}(V_B + V_A) = Sd\theta \cdot V_{AB}(V_B - \tfrac{1}{2}V_{AB}) = Td\theta \\ &= K\theta\, d\theta \end{aligned}$$

since $V_B = V_A + V_{AB}$, and since $Td\theta = K\theta\, d\theta$ is the work done in twisting the bifilar (or other) suspension through the angle $d\theta$ by the torque T of the electrical forces, K being the constant of torsion of the suspension. The last equation gives

$$T/K = \theta = \tfrac{1}{2}S/K \cdot V_{AB}(V_B + V_A) = S/K \cdot V_{AB}(V_B - \tfrac{1}{2}V_{AB}) \quad (10)$$

(1) If V_A and $V_B = V_A + V_{AB}$ are very large in comparison with V_{AB}, the voltage to be measured, V_{AB} may be neglected without appreciable error in the expression $(V_B - \tfrac{1}{2}V_{AB})$, and θ is sensibly proportional to V_{AB} and to V_B. Hence by making V_B large, even small potential differences V_{AB} may be measured with accuracy. In this case (10) becomes

$$V_{AB} = K/SV_B \cdot \theta \quad (11)$$

(2) If the needle is in metallic contact with one of the quadrant pairs, as AA', $V_A = 0$, $V_B = V_{AB}$, and (10) becomes

$$V_{AB}^2 = 2K/S \cdot \theta \quad (12)$$

Since the deflection in this case is proportional to the square of the voltage, alternating as well as steady voltages can be measured; but low voltages, either steady or alternating, cannot be measured with accuracy (except with very sensitive instruments).

(3) The quadrant pairs AA' and BB' are connected to the terminal plates of an auxiliary voltaic battery consisting of an even number of similar cells in series, and one pole of the voltaic cell

or other agent whose voltage V is to be measured is connected to the needle C, the other to the central point of the auxiliary battery. Then, if Ψ denotes the e.m.f. of the auxiliary battery, $V_{AB} = \Psi$, $V_A = V - \frac{1}{2}\Psi$, $V_B = V + \frac{1}{2}\Psi$; and (10) becomes

$$V = K/S\Psi \cdot \theta \qquad (13)$$

In this arrangement the deflection is *accurately* proportional to V, whether V is large or small in comparison with Ψ.

The first and third methods of using the instrument, in which a supplementary voltage is employed in addition to that to be measured, are called *heterostatic* methods; the second is called *idiostatic*.

While the quadrant electrometer cannot be used for *absolute* measurements, the factor multiplying θ being impossible to determine with accuracy *directly*, this factor can be determined in any case by measuring the deflection produced by a *known* voltage, such as that of a standard cell.

7. Condensers in Multiple. When any number n of condensers whose separate capacities are S_1, S_2, $\cdots$, S_n are connected in

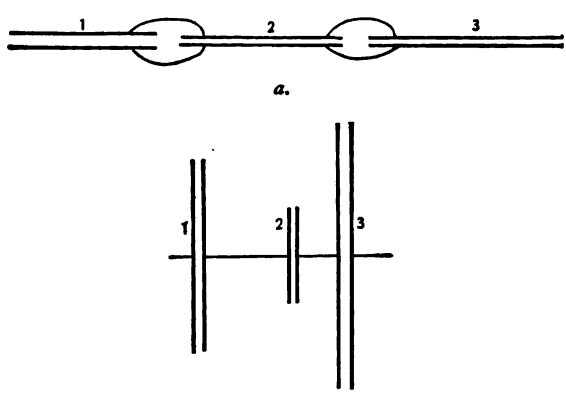


Fig. 53.

multiple, as in Fig. 53, *a*, a compound condenser is formed whose capacity is

$$S = S_1 + S_2 + \cdots + S_n \qquad (14)$$

provided that the field of each condenser is included, practically, between its plates only, and therefore does not affect appreciably the fields of the other condensers.

For if V is the common voltage between the separate pairs of plates of the compound condenser, q the total charge on each compound plate, and $q_1, q_2, \cdots, q_n$ the charges on the separate plates when the condensers are charged separately to the voltage V, we have, for such a system,

$$q = q_1 + q_2 + \cdots + q_n \qquad \text{and}$$

$$S = q/V = (q_1 + q_2 + \cdots + q_n)/V = (S_1V + S_2V + \cdots + S_nV)/V$$
$$= S_1 + S_2 + \cdots + S_n$$

8. Condensers in Series. When n condensers of individual capacities S_1, S_2, etc., are connected up in series, as in Fig. 53, *b*, a compound condenser is formed of capacity

$$S = 1/(1/S_1 + 1/S_2 + \cdots + 1/S_n) \tag{15}$$

provided that the plates of each condenser are so close together that sensibly all the tubes from one plate terminate upon the other.

For if V is the total potential difference between the terminal plates of the compound condenser, q the (numerical) charge on each of them, and q_1, q_2, etc., and V_1, V_2, etc., the charges and voltages of the individual condensers, we have

$$q = q_1 = q_2 = q_3 = \cdots = q_n$$

since the intermediate plates are all charged by induction, and sensibly all the tubes from one plate of each condenser terminate upon the other.

Also $$V = V_1 + V_2 + \cdots + V_n$$

Hence

$$S = q/V = q/(V_1 + V_2 + \cdots + V_n) = q/(q/S_1 + q/S_2 + \cdots + q/S_n)$$

from which (15) follows on cancelling q.

9. Some Electrostatic Methods of Comparing Capacities. In each of the following methods the capacity of the electrometer, or electrometers, and connecting wires is supposed to be negligible in comparison with the capacities to be compared, or else to be included with them.

(1) The capacities S_1 and S_2 to be compared are connected in series with a battery of electromotive force V, and an electrometer is connected across the plates of each; or an electrometer is connected across the plates of one, for example S_2, and another, with the battery, across the terminal plates. In the first case we have

$$S_1 V_1 = S_2 V_2$$

whence

$$S_2/S_1 = V_1/V_2 \qquad (16)$$

and in the second case

$$S_2 V_2 = V/(1/S_1 + 1/S_2)$$

whence

$$S_2/S_1 = V/V_2 - 1 \qquad (17)$$

If the leakage and absorption (Chapter VI.) of the condensers are negligible, the two measurements may be made in succession with a single electrometer.

(2) The capacities to be compared are arranged to be put in multiple by a switch K. With K open let S_1, to whose plates the quadrants of an electrometer are connected, be charged to a voltage V, and then connected in multiple with S_2, when both condensers will come to voltage V_2. Then we have

$$S_1 V = (S_1 + S_2) V_2$$

whence

$$S_2/S_1 = V/V_2 - 1 \qquad (18)$$

(3) In this method the condensers whose capacities S_1 and S_2 are to be compared are charged in multiple to the voltage V, insulated, and then again connected in multiple, but in such a way that the positive and negative plates of 1 are connected to

the negative and positive plates of 2, the final voltage being V_2. Immediately after charging

$$q_1 = S_1 V \quad \text{and} \quad q_2 = S_2 V$$

After the final connection in multiple

$$q_1 - q_2 = (S_1 - S_2) V = (S_1 + S_2) V_2$$

Hence

$$S_2 / S_1 = (V - V_2)/(V + V_2) \tag{19}$$

It is obvious that the above three methods cannot be applied when one or both of the condensers are of the guard ring type, thus having more than two conductors. The following method of testing the equality of the capacities of two guard ring condensers was devised by Maxwell. It can also be applied when only one, or neither, of the condensers is of the guard ring type. In the last case it becomes identical with the last of the preceding methods, which is an extension of a method due to Cavendish.

10. Maxwell's Method of Testing the Equality of the Capacity of a Guard Ring Condenser and that of any Other Condenser.* Let A be the disk, B the guard ring and sheath, and C the larger plate of one of the condensers; and let A', B', and C' be the corresponding parts of the other. If either condenser, as ABC, is of the simpler form with only two conductors, we have only to suppress B and to suppose A and C to be the two conductors, it being understood that sensibly all the tubes of induction pass from one plate to the other when the condenser is charged.

Let B be kept always connected with C', and B' with C. Then

(1) Let A be connected with B, and C' with J, the positive (for the sake of definiteness) terminal of a battery or other source of electrification, the other terminal of which is connected to

* Maxwell, *Treatise*, § 229.

earth; and let A' be connected with B' and C and with the earth. The two condensers are now charged oppositely, so that A is positive and A' negative, and the field of each is sensibly confined to the region between the plates.

(2) Let A, B, and C' be insulated from J.

(3) Let A be insulated from B and C', and A' from B' and C.

(4) Let B and C' be connected with B' and C and with the earth. The charges on A and A' remain unaltered in magnitude, but are now distributed over their whole surfaces, the fields no longer being confined to the regions between the plates.

(5) Let A be connected with A'.

(6) Let A and A' be connected with one quadrant pair of an electrometer E, the other quadrants of which are earthed. If the charges of A and A' are equal in magnitude, the electrification wholly disappears, since they have opposite signs, and the electrometer is unaffected. In this case the fields connected with A and A' have the same capacities. Otherwise, the electrometer will indicate positive or negative electrification according as A or A' has the greater charge and therefore the greater capacity.

By making repeated tests and adjustments, if necessary, the capacity of a condenser constructed with movable conductors so as to have a variable capacity, or a condenser in the process of construction, may be made equal to that of a standard condenser of the guard ring form.

Other methods of comparing capacities are described in Chapters XII. and XIII.

11. Some Methods of Extending the Range of an Electrometer.* If the ratios of the capacities of the condensers in the three first arrangements described above are known, the three methods of comparing capacities may be inverted for the measurement of

* Cf. Maxwell, *Treatise*, § 220; Lord Kelvin, *B. A. Report*, 1885, p. 907.

high voltages with electrometers constructed for low voltage measurement. Thus we have, from (17), (18) and (19),

$$\left.\begin{array}{ll}(1) & V=[(S_2+S_1)/S_1]\,V_2 \\ (2) & V=[(S_2+S_1)/S_1]\,V_2 \\ (3) & V=[(S_1+S_2)/(S_1-S_2)]\,V_2\end{array}\right\} \qquad (20)$$

By measuring V_2, V may be determined; so that by properly choosing or adjusting the ratio of the capacities the range of an electrometer may be almost indefinitely extended.

CHAPTER IV.

GENERAL ELECTROSTATIC THEORY. IDEAL FIELDS CONTAINING TWO OR MORE DIELECTRICS.

1. Generalisation of Gauss's Theorem.* In § 23, Chapter I., this theorem was established for a surface enclosing a single homogeneous isotropic dielectric, or such a dielectric and conductors. We shall now show that it holds for a closed surface cutting any number of such dielectrics, or such dielectrics and conductors. To do this it is necessary to show only that the strength of a tube of induction is not altered when it passes from one dielectric into another.

For this purpose, consider the electric field between the plates A and B of a closed condenser containing two dielectrics 1 and 2, 1 being in contact with A only, and 2 in contact with B only. If the charge of A is q, that of B is $-q$, and there is no charge upon the interface between the dielectrics 1 and 2. (If there are charges due to contact, they are equal and opposite at any point of the interface.) Applying Gauss's theorem to the region 1, we find the total strength of all the tubes emanating from A to be q; and, likewise, in the region 2, the total strength of all the tubes terminating upon B to be q. Thus the total strength of all the tubes is unchanged in passing across the interface from A to B. And since this result is absolutely independent of the size or shape of the dielectrics, that is of the shapes of the tubes, it follows that the strength of *every* tube remains constant in crossing the interface, howsoever the field is divided up into tubes.

It may be shown that the theorem is also valid in the general case when the dielectrics are neither homogeneous nor isotropic,

* See *The Physical Review*, September, 1902, p. 173.

but the demonstration lies outside the scope of this work. In all that follows we shall assume the theorem to be perfectly general.

2. The (Uncharged) Interface Between Two Dielectrics. Laws of Refraction of Lines of Intensity and Displacement. At an uncharged interface *S*, Fig. 54, between two dielectrics 1 and 2 with permittivities c_1 and c_2, certain conditions, which we proceed to find, must be satisfied by the electric intensity and displacement.

In the first place, the line integral of the electric intensity around the infinitesimal circuit *adfca*, in which *ad* and *cf* are par-

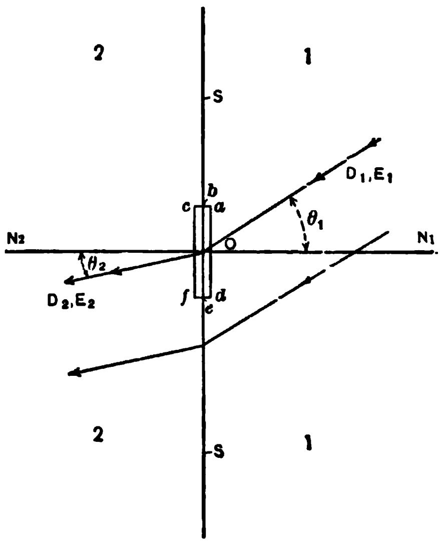


Fig. 54.

allel to the interface, and *ac* and *df* normal to the interface, is zero, since the field is static. This integral, that is, the e.m.f. around the circuit, is

$$E_1 \sin\theta_1\, ad + E_1 \cos\theta_1\, de + E_2 \cos\theta_2\, ef + E_2 \sin\theta_2\, fc + E_2 \cos\theta_2\, cb + E_1 \cos\theta_1\, ba = 0$$

But $de = -ba$, and $ef = -cb$, hence all the terms but the first and fourth cancel, leaving

$$E_1 \sin\theta_1\, ad + E_2 \sin\theta_2\, fc = 0$$

or, since $fc = -ad$,

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \qquad (1)$$

Thus the tangential component of the intensity does not change on crossing the interface.

Moreover, E_2 lies in the plane containing E_1 and the normal to the interface, N_1N_2. For if E_2 were not in this plane, it would have a component perpendicular to this plane, while E_1 has no such component. Therefore the e.m.f. around a circuit in a plane perpendicular to the interface and lying partly in medium 1 and partly in medium 2 would differ from zero, which is impossible in the static field.

Consider finally the electric flux outward across the surface of an elementary parallelepiped *acfd* with center at o in the interface; two of the faces, of breadth *ad* and height *h* (perpendicular to the paper) being parallel to *S*, and the others, of breadth *ac* and height *h*, normal to *S*. This flux must be zero, by Gauss's theorem. Hence

$$-D_1 \sin\theta_1\, ab\, h - D_1 \cos\theta_1\, ad\, h + D_1 \sin\theta_1\, de\, h$$
$$+ D_2 \sin\theta_2\, ef\, h + D_2 \cos\theta_2\, cf\, h - D_2 \sin\theta_2\, bc\, h = 0$$

But since $de = ef = ab = bc$, and $ad = cf$, this reduces to

$$D_2 \cos\theta_2 - D_1 \cos\theta_1 = 0$$

or

$$\left.\begin{aligned} D_1 \cos\theta_1 &= D_2 \cos\theta_2 \\ c_1 E_1 \cos\theta_1 &= c_2 E_2 \cos\theta_2 \end{aligned}\right\} \qquad (2)$$

Thus the normal component of the displacement does not change in crossing the interface.

From (1) and (2) we have by division,

$$\tan\theta_1 / \tan\theta_2 = (D_1/E_1)/(D_2/E_2) = c_1/c_2 \qquad (3)$$

In passing from one dielectric to another lines of displacement are therefore refracted in such a way that

I. The incident and refracted lines are in the same plane perpendicular to the interface at the point of incidence; and that

II. The ratio of the tangent of the angle of incidence to the tangent of the angle of refraction is a constant for the given media, and equal to the ratio of the two permittivities.

Since $\tan\theta_1$ and $\tan\theta_2$ become infinite together when

$$\theta_1 = \theta_2 = 90^\circ$$

no phenomenon similar to total reflection in optics occurs.

If θ_2 is kept constant, and c_2/c_1 diminished, ... In the limit when $c_2/c_1 = 0$, $\tan\theta_1 =$ infinity, a... In this limiting case θ_2 is of course meaningless; sinc... $_2/c_1 = 0$, there is no electric field in medium 2. As st... 14, I., no substance has a permittivity less than $c_0 = 1$, but for the sake of certain analogies (VIII. and IX.) the imaginary case of $c_2/c_1 = 0$ is here considered.

If while θ_2 is kept constant, c_2/c_1 is increased, θ_1 increases, approaching 0 as c_2 approaches infinity. In this limit also θ_2 is meaningless, and medium 2 contains no electric field, as D would there be infinite if E were greater than zero. Since in a static field the lines of intensity always meet the surface of a conductor normally ($\theta_1 = 0$) and since there is no electric field within the conductor, a conductor behaves in a static field like a substance of infinite permittivity. Since in this case the displacement is discontinuous at the surface, the conductor's surface is charged. This behavior, however, is not due to the conductor's permittivity, but to its conductivity. Of the permittivity of most conductors little is known.

An experimental method of verifying (3) is described in § 5, VII.

3. Fictitious or Apparent Electric Charges. The discontinuity in the normal component of the electric intensity at any point of

the interface, viz., $E_1 \cos \theta_1 - E_2 \cos \theta_2$ (E_1 and E_2 being reckoned positive when directed *from* medium 2 *to* medium 1), is exactly the same as it would be if c_2 were equal to c_1 and there were a discontinuity in the normal component of the displacement at the point equal to $c_1 (E_1 \cos \theta_1 - E_2 \cos \theta_2)$. This would leave E_1 and E_2 everywhere unaltered, and would leave D_1 unaltered; but since D_2 would now equal $c_1 E_2$ instead of $c_2 E_2$, as before, it would decrease D_2, and therefore all the charges in medium 2, in the ratio c_1/c_2.

Thus the electric intensity everywhere in the field containing two dielectrics in contact (the interface being uncharged) is the [illegible] uld be if medium 2 were replaced by medium 1, if th[illegible] interface were charged to a surface density $\sigma' = c_1$[illegible] $- E_2 \cos \theta_2$), and if all the charges in medium 2 (or at th[illegible]s, if any, between medium 2 and conductors, which could be replaced by medium 2 (§ 28, I.) without altering the field) were reduced in the ratio c_1/c_2. Imagining these changes made in any case, we can compute the intensity at any point by the direct application of the law of inverse squares. From the intensity and the permittivity at any point the displacement can be found, and from the charges and the intensity the mechanical forces upon the charged bodies. This is an extension of the method of § 28, I., which treats of the case in which $c_2 =$ infinity, or $c_1/c_2 = 0$, only. The mechanical force at the interface between the two dielectrics will be determined in §§ 6 and 9.

The quantity $$\sigma' = c_1(E_1 \cos \theta_1 - E_2 \cos \theta_2) \qquad (4)$$

is called the *apparent* or *fictitious electric surface density* at the point with respect to medium 1. [In the irrational systems of units, Chapter XIV., σ' is defined by the equation

$$4\pi\sigma' = c_1(E_1 \cos \theta_1 - E_2 \cos \theta_2)]$$

By simply interchanging the subscripts 1 and 2 we could of course refer everything to the dielectric 2. In all that follows,

however, the medium designated as 1 will be taken as the standard medium, and the apparent charges, etc., will be computed with respect to it.

If there are several dielectrics in the field, it can be shown by the method used above that, to reduce everything to medium 1 for the sake of computing the electric intensity by the law of inverse squares, a surface density must be assumed at every point of each interface equal to $c_1 \times$ the normal discontinuity of E at the point, and the charges in any medium of permittivity c must be reduced in the ratio c_1/c.

In the same way, if the permittivity varies continuously, instead of suddenly at distinct interfaces, there will be an *apparent volume density* of electrification equal, at a point where the intensity is E, to

$$\rho' = c_1 \operatorname{div} E \tag{5}$$

(In the irrational systems of units $4\pi\rho' = c_1 \operatorname{div} E$.)

If both volume density and surface density of apparent electrification are present, we have

$$q' = \int \sigma' dS + \int \rho' d\tau \tag{6}$$

the first integral being extended over all fictitiously charged surfaces, and the second throughout all fictitiously electrified volumes.

Electric Poles. The fictitiously charged surfaces or volumes, that is, the surfaces or volumes where the electric intensity is discontinuous, are called *electric poles*. The total apparent charge in any region is the *strength* of the pole (or portion of a pole) in that region, and is equal to $c_1 \times$ the outward flux of the electric intensity * across a closed surface surrounding the pole, by (4) and (5). Another expression is given below. The pole is *positive* or *negative* according as the apparent charge is positive or negative.

* The flux of any vector across a surface is the integral over the surface of the normal component of the vector.

4. Fictitious Charges (continued). Intensity of Electrisation. Electric Susceptibility. (4) may be written

$$\sigma' = c_1(E_1 \cos\theta_1 - E_2 \cos\theta_2) = (D_2 - c_1E_2)\cos\theta_2 \qquad (7)$$

The quantity $D_2 - c_1E_2$, the difference between the actual displacement in medium 2 and the displacement which would exist there with the same value of intensity if c_2 were equal to c_1, is called the *intensity of electrisation* of medium 2 with respect to medium 1, and is denoted by J. Thus

$$J = D_2 - c_1E_2 \qquad (8)$$

Another definition of J is given in § 12.

(In the irrational systems of units, Chapter XIV., J is defined by the relation $4\pi J = D_2 - c_1E_2$.)

When none of the electrisation is intrinsic (§ 1, VI.), (8) may be written

$$J = D_2 - c_1E_2 = [(c_2 - c_1)/c_2]D_2 \qquad (9)$$

J is evidently a vector with the same direction as that of D_2 or the opposite direction, according as D_2 is greater or less than c_1E_2; or, when (9) is valid, according as c_2 is greater or less than c_1.

(9) may be written

$$J = (c_2 - c_1)E_2 = \kappa E_2 \qquad (10)$$

$\kappa = (c_2 - c_1)$ is called the *electric susceptibility* of medium 2 with respect to medium 1. [In the irrational systems of units,

$$\kappa = (c_2 - c_1)/4\pi]$$

(8) and (10) may be transformed into

$$D_2 = c_2E_2 = J + c_1E_2 = (c_1 + \kappa)E_2 \qquad (11)$$

In terms of J, the apparent surface density is

$$\sigma' = J\cos\theta_2 \qquad (12)$$

and the apparent charge upon a surface is

$$q' = \int J \cos\theta_2 dS \qquad (13)$$

The volume density of fictitious electrification is

$$\rho' = c_1 \operatorname{div} E = \operatorname{div}(D - J) = \operatorname{div} D - \operatorname{div} J = -\operatorname{div} J = \operatorname{conv} J \quad (14)$$

since $\operatorname{div} D = 0$.

The total apparent charge within a volume τ is

$$q' = \int \rho' d\tau = \int \operatorname{conv} J\, d\tau \quad (15)$$

The total apparent charge in a pole distributed over a surface S and through a volume τ is

$$q' = \int J \cos\theta_2 dS + \int \operatorname{conv} J\, d\tau \quad (16)$$

The total apparent charge within the volume τ and upon the surface S of a dielectric 2 completely surrounded by a homogeneous dielectric 1 is zero. This may be proved by integrating the first term of (16) over the whole interface and the second term throughout the whole volume of the dielectric 2. The equation may be written

$$q' = \int D_2 \cos\theta_2 dS - c_1 \int E_2 \cos\theta_2\, dS - \int \operatorname{div} D_2 d\tau + c_1 \int \operatorname{div} E_2 d\tau$$

The first and third terms are evidently zero, and it will be shown that the second and fourth terms cancel. For $\operatorname{div} E_2\, d\tau$ is the excess of the flux of intensity leaving the volume $d\tau$ over that entering the volume $d\tau$. Hence $\int \operatorname{div} E_2\, d\tau$ throughout the volume τ is equal to the total excess of the flux of intensity leaving the whole volume over that entering the same volume; and this is equal and opposite to $-c_1 \int E_2 \cos\theta_2 dS$, which is the excess of the intensity flux entering over that leaving the whole volume. Thus the proposition is established.

A dielectric in which

$$\rho' = \operatorname{conv} J = c_1 \operatorname{div} E = 0$$

is said to possess *solenoidal* electrisation for the reason that in this case all the tubes, or *solenoids*, of intensity (E) or electrisation (J) run through the dielectric from one pole face to the other without discontinuity at fictitious charges between.

A medium in which J or D, together with c, is constant is said to be *uniformly electrised.*

5. General Expression for the Potential at a Point ($\int E\cos\theta dL$ from the point to *infinity*). When fictitious charges are present, we must, to find the potential at a point, suppose all the true charges reduced in the ratio c_1/c, and add to the expression for the potential due to the true charges alone, (16), II., a term $\int dq'/4\pi c_1 L$. Thus, in the most general case,

$$V=1/4\pi c_1\cdot(\int c_1/c\cdot dq/L+\int dq'/L)=1/4\pi\cdot\int(dq/cL+dq'/c_1L) \quad (17)$$

where c is the permittivity at the seat of the true charge dq. Or, if we call $c_1/c\cdot dq$ also an apparent charge, we have, instead of (17),

$$V=1/4\pi c_1\cdot\int dq'/L \quad (18)$$

6. The Integral Force Upon an Electric Pole. The electric intensity E' at a point P "due to" an electric pole of strength q' is

$$E'=\int dq'/4\pi c_1 L^2 \quad (19)$$

where L is the distance from the seat of dq' to P, and the integration is a vector integration, the direction as well as the magnitude of L being different for each different element dq'.

The force upon a small body at P with a concentrated *true* charge q is

$$E'q=\int q/4\pi c_1 L^2\cdot dq'=\int E\,dq'$$

where E is the intensity at the seat of dq' due to the charge q.

Thus the *total* force F upon an electric pole is

$$F=\int E\,dq' \quad (20)$$

where E is the intensity at the seat of dq' due to the other poles and true charges, the integration being a vector integration.

The traction per unit area at any *element* of an interface is computed in § 9.

The force between two concentrated poles with apparent charges q' and q'' distant L apart in a dielectric of permittivity c would be

$$F = q'q''/4\pi c L^2 \tag{21}$$

7. The Infinite Parallel Plate Condenser with Two or More Dielectrics in the Form of Infinite Plane Slabs. The simplest possible field involving two dielectrics is that of an infinite parallel plate condenser with two dielectrics 1 and 2 of permittivities c_1

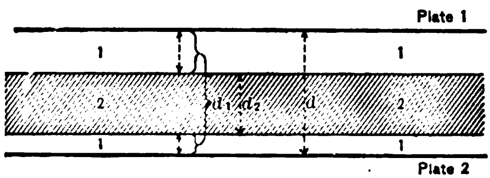


Fig. 55.

and c_2 in the form of infinite plane slabs of thicknesses d_1 and d_2 parallel to the condenser plates distant $d = d_1 + d_2$ apart (Fig. 55).

Here the tubes of induction evidently run straight across without change of strength from one condenser plate to the other, meeting both conductors normally. If $D_1 = D_2 = D$ denotes the displacement, the intensities in media 1 and 2 are

$$E_1 = D_1/c_1 = D/c_1 \text{ and } E_2 = D_2/c_2 = D/c_2$$

respectively. Hence the voltage of the condenser is

$$V_{12} = E_1 d_1 + E_2 d_2 = D/c_1 \cdot \{d - [(c_2 - c_1)/c_2] d_2\}$$

The capacity of a right prism of the dielectrics of thickness d and cross-section A is

$$S = A\,D/V_{12} = Ac/\{d - [(c_2 - c_1)/c_2] d_2\} \tag{22}$$

and the energy contained in the prism is

$$\begin{aligned} W &= \tfrac{1}{2} A D V_{12} = \tfrac{1}{2} A D^2/c_1 \cdot \{d - [(c_2 - c_1)/c_2] d_2\} \\ &= \tfrac{1}{2} A c_1 V_{12}^2/\{d - [(c_2 - c_1)/c_2] d_2\} \end{aligned} \tag{23}$$

Thus the substitution of dielectric 2 for a portion of dielectric 1 (cf. § 51, Chapter I.) decreases the energy if the charges are kept the same, and increases the energy if the voltage is kept the same, provided c_2 is greater than c_1. If c_2 is less than c_1, the opposite is true.

If the second dielectric does not touch either condenser plate, the force upon either plate due to the discontinuity of the displacement, viz., $\frac{1}{2}E_1D$ per unit area, is not altered by its introduction when D is kept the same; but if V_{12} is kept the same, the force upon the area A becomes

$$F = \tfrac{1}{2}Ac_1V_{12}^2/\{d-[(c_2-c_1)/c_2]d_2\}^2 \qquad (24)$$

which is greater or less than when the whole dielectric had the permittivity c_1 according as c_2 is greater or less than c_1.

If the dielectric 2 is in contact with plate 2, and the displacement D as before, the force per unit area upon plate 1 is, as before,

$$f_1 = \tfrac{1}{2}E_1D = \tfrac{1}{2}D^2/c_1$$

but the force per unit area upon plate 2, due to the discontinuity of the displacement at its surface, is

$$f_2 = \tfrac{1}{2}E_2D = \tfrac{1}{2}D^2/c_2$$

which is less than f_1 if c_2 is greater than c_1. But the tension along the lines of intensity in medium 1 is $\frac{1}{2}E_1D$, and in medium 2, $\frac{1}{2}E_2D$. Hence there is a mechanical force upon dielectric 2 acting toward plate 1 of magnitude, per unit area,

$$f_3 = \tfrac{1}{2}E_1D - \tfrac{1}{2}E_2D = \tfrac{1}{2}D^2(c_2-c_1)/c_1c_2 \qquad (25)$$

and this force is transmitted mechanically by the dielectric to plate 2, making the total force per unit area upon the plate equal to

$$f_2 + f_3 = \tfrac{1}{2}D^2/c_2 + \tfrac{1}{2}D^2(c_2-c_1)/c_1c_2 = \tfrac{1}{2}D^2/c_1 = f_1$$

The apparent surface density at the interface is uniform and equal to

$$\sigma' = D(c_2-c_1)/c_2 = J \qquad (26)$$

The electrisation is solenoidal.

When the number of dielectrics is greater than two, the intensity, fictitious charges, etc., can easily be determined by the same method.

8. The Spherically and Cylindrically Radial Fields. The field surrounding a point charge at the center of any number of concentric spherical shells of different permittivities, or the field of a spherical condenser with any number of dielectrics in the form of concentric spherical shells, together with the fictitious charges, etc., as well as the cylindrically radial field in coaxial cylindrical shells of dielectric, can be easily found by the method of the foregoing article, *i. e.*, by the direct application of Gauss's theorem.

As an example, suppose we have a spherical field in three dielectrics, the charge, q, being in medium 3, and medium 1 surrounding media 2 and 3 and extending to infinity, Fig. 56. The

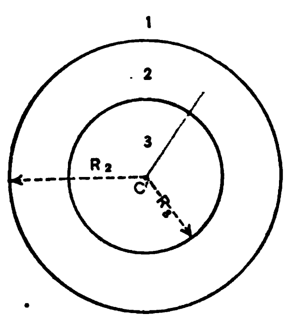


Fig. 56.

displacement at any point distant R from C is $q/4\pi c R^2$, and the intensity is equal to the displacement divided by the permittivity at the point.

We shall also find the intensity by means of the fictitious charges. It can be computed by considering a charge

$$q_3' = q c_1/c_3$$

to exist at C instead of q; a uniformly distributed charge

$$q_{23}' = 4\pi R_3^2 c_1(q/4\pi c_2 R_3^2 - q/4\pi c_3 R_3^2) = qc_1[(c_3 - c_2)/c_2 c_3]$$

at the interface 23; and a uniformly distributed charge

$$q_{21}' = qc_1[(c_2 - c_1)/c_1 c_2] = q[(c_2 - c_1)/c_2]$$

at the interface 21; all in a dielectric of permittivity c_1. Each charge "produces" outside the surface on which it is distributed the same effect as if it were concentrated at the center C, and within the surface no effect at all. Thus the intensity at a point distant R from C, when R is greater than R_2, is

$$E = (q_3' + q_{23}' + q_{21}')/4\pi c_1 R^2 = q/4\pi c_1 R^2 \qquad (27)$$

while the intensity at a point in medium 2 distant from the center c by R, less than R_2 and greater than R_3, is

$$E = (q_3' + q_{23}')/4\pi c_1 R^2 = q/4\pi c_2 R^2 \qquad (28)$$

9. The Mechanical Force at the Uncharged Interface Between two Dielectrics. In the particular case considered in § 7, where the lines of induction were normal to the interface, there was

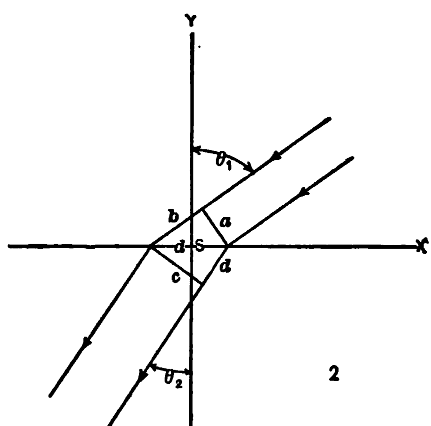


Fig. 57.

found to be a force at the interface normal to it and equal, per unit area, to $\frac{1}{2}c_1E_1^2 - \frac{1}{2}c_2E_2^2$ measured in the direction 21. In this article we shall find the force per unit area at any point P of the interface in the general case when the lines of induction make

any angles (Figs. 54 and 57), connected by the relations (1) and (2) with the normal at P to the interface.

In Fig. 57 let the plane XY (the plane of the paper) coincide with the plane containing E_1 and E_2 at the given point of the interface, the axis Z being perpendicular to XY through P and the plane of the interface coinciding with the plane XZ. Let $dS = dxdz$ be a rectangular element of area of the interface at the point, with its sides parallel to X and Z respectfully. We shall find the components dX and dY of the force upon dS in the directions X and Y, parallel and perpendicular, respectively, to the interface. It is evident, from symmetry, that the component in the direction Z is zero.

The force upon dS is clearly equal to the force upon the surface $abcd$ formed by drawing rectangles a, b, c, d, all of breadth dz, through the ends of dS, with their planes parallel and perpendicular to E_1 and E_2. The areas of these rectangles are

$$a = dS\cos\theta_1,\quad b = dS\sin\theta_1,\quad c = dS\cos\theta_2,\quad d = dS\sin\theta_2$$

Let T_1 and $p_1 = T_1 = \frac{1}{2}c_1E_1^2$ denote the tension and pressure parallel and perpendicular, respectively, to the intensity in medium 1, and T_2 and $p_2 = T_2 = \frac{1}{2}c_2E_2^2$ the corresponding quantities in medium 2.

The force upon the face a is $T_1 \cdot a = T_1dS\cos\theta_1$, with the components

$$dX_a = T_1dS\cos\theta_1\cdot\sin\theta_1 \quad\text{and}\quad dY_a = T_1dS\cos\theta_1\cdot\cos\theta_1$$

in the positive directions of X and Y. The force upon the face b is $p_1\cdot b = p_1dS\sin\theta_1$, with the components

$$dX_b = p_1dS\sin\theta_1\cdot\cos\theta_1 \text{ and } dY_b = -p_1dS\sin\theta_1\cdot\sin\theta_1$$

in the positive directions of X and Y. In like manner, the components of the force upon the faces c and d are

$$dX_c = -T_2dS\cos\theta_2\cdot\sin\theta_2,\quad dY_c = -T_2dS\cos\theta_2\cdot\cos\theta_2$$

and $dX_d = -p_2dS\sin\theta_2\cdot\cos\theta_2,\quad dY_d = p_2dS\sin\theta_2\cdot\sin\theta_2$

Hence we have, for the X and Y components of the total force upon dS,

$$dX = dX_a + dX_b + dX_c + dX_d = [\tfrac{1}{2}c_1E_1^2(2\sin\theta_1\cos\theta_1) - \tfrac{1}{2}c_2E_2^2(2\sin\theta_2\cos\theta_2)]\,dS$$

and

$$dY = dY_a + dY_b + dY_c + dY_d = [\tfrac{1}{2}c_1E_1^2(\cos^2\theta_1 - \sin^2\theta_1) - \tfrac{1}{2}c_2E_2^2(\cos^2\theta_2 - \sin^2\theta_2)]\,dS$$

The X and Y components of the force per unit area upon the interface at P are thereforc

$$\begin{aligned} dX/dS &= \tfrac{1}{2}c_1E_1^2 2\sin\theta_1\cos\theta_1 - \tfrac{1}{2}c_2E_2^2\, 2\sin\theta_2\cos\theta_2 \\ &= \tfrac{1}{2}c_1E_1^2\sin 2\theta_1 - \tfrac{1}{2}c_2E_2^2\sin 2\theta_2 \end{aligned} \qquad (29)$$

and

$$\begin{aligned} dY/dS &= \tfrac{1}{2}c_1E_1^2(\cos^2\theta_1 - \sin^2\theta_1) - \tfrac{1}{2}c_2E_2^2(\cos^2\theta_2 - \sin^2\theta_2) \\ &= \tfrac{1}{2}c_1E_1^2\cos 2\theta_1 - \tfrac{1}{2}c_2E_2^2\cos 2\theta_2 \end{aligned} \qquad (30)$$

On multiplying together equations (1) and (2), we find

$$c_1E_1^2\sin 2\theta_1 = c_2E_2^2\sin 2\theta_2 \qquad (31)$$

Hence

$$dX/dS = dZ/dS = 0 \qquad (32)$$

and the total force at the interface is normal to the surface and equal to dY/dS.

By making use of (31), (30) may also be written

$$dY/dS = \tfrac{1}{2}c_2E_2^2\sin 2(\theta_2 - \theta_1)/\sin 2\theta_1 \qquad (33)$$

When c_2 is greater than c_1, $\theta_2 - \theta_1$ is positive by (3), and dY/dS is positive, that is, directed toward medium 1.

When $\theta_2 = \theta_1 = 0°$, (30) and (33) reduce to (25).

10. The Process of Changing the Dielectric Within the Plates of an Ordinary Parallel Plate Condenser is of much interest. If the plates have charges q and $-q$, q unit tubes will pass from one

to the other. When the permittivity of the medium inside the plates is the same as that of the medium outside, c_1, the tubes will have some such distribution as that indicated in Fig. 47.

If a slab of a dielectric of greater permittivity, c_2, is introduced between the plates, the tubes will crowd into this dielectric until (3) is satisfied, leaving fewer tubes in the region outside than before. That this takes place follows from the consideration that if the induction between the plates were not to increase, the lateral pressure in that region, which is proportional to D^2/c, would be insufficient to maintain equilibrium, c_2 being greater than c_1, and equilibrium having existed when c_2 was equal to c_1, or when the dielectrics outside and inside were the same. Since the dielectric outside is unaltered, we can compare the voltages of the condenser in the two cases conveniently, if we measure them along the same path *outside* before and after the introduction of the slab. Since the field in the external region is weaker after the introduction than before, the voltage is seen to be less. If c_1 is greater than c_2, the effects are of course opposite.

To look at the matter in another way, the tubes connecting the outside surfaces and those connecting the inside surfaces may be regarded as the tubes of two condensers connected in parallel. If the capacity of either is increased by increasing the permittivity of its dielectric, the tubes will crowd into that one, and the common voltage of both will be reduced. If the distribution of the tubes remained unaltered, the voltage between the two plates would be greater along a line not passing through the slab than along a line passing through the slab.

During the lateral introduction of the slab into the region between the condenser plates, the tubes crowding into it exert a pull upon it, by § 9, which continues until it is symmetrically situated with respect to the plates, when the pulls urging it in all directions balance. If during the change the charges are kept constant, the energy decreases, since the voltage decreases; if the voltage is kept constant the charges increase and the energy

increases. By computing the space rate of this increase or decrease of the energy, the force acting upon the slab may be found by the method of § 55, Chapter I. This computation is made in § 4, Chapter VII. The force could not, in general, be determined without very great difficulty by the method of § 9.

11. Field Surrounding a Concentrated Charge Situated in One of Two Infinite Dielectrics Separated by a Plane Interface. Let the charge q be concentrated at the point A, Fig. 58, in the me-

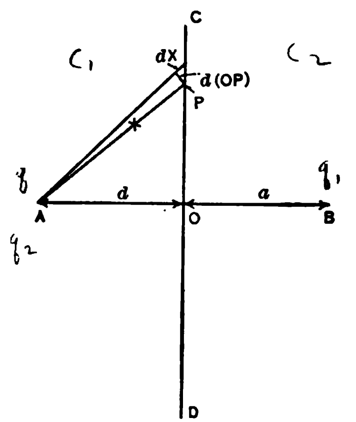


Fig. 58.

dium of permittivity c_1 distant d from the interface separating the medium of permittivity c_1 from that of permittivity c_2.

By § 48, I., there is only one field which can satisfy the given conditions. The given equipotential in this case is the infinite sphere at zero potential with A as center. To find the field by means of the law of inverse squares, we must reduce the problem to one with a single dielectric, until the displacement, or else the intensity, everywhere is found. Then the unknown one of the two can be found in each dielectric from the relation $D = cE$. We shall combine the method of images with the method of § 3.

Guided by the result of § 15, II., which solves the problem when c_2 = infinity, and by what we have learned of the refraction of lines of displacement, the simplest rational assumption we can

make is that to the left of CD the displacement is such as would accompany a charge q at A and a charge q_1 at B, both in a dielectric of permittivity c_1; and that the displacement to the right of CD is such as would emanate radially in the medium 1 from a charge q_2 at A. If q_1 and q_2 can be given such values as to satisfy (1) and (2), at every point of CD, the assumption will be justified and the problem solved.

Choosing OB and OC as positive directions, we find, for the normal displacement at P on the left side of CD,

$$qd/4\pi x^3 - q_1 d/4\pi x^3 = (q - q_1)d/4\pi x^3$$

and for that on the right side

$$q_2 d/4\pi x^3$$

(2) Will be satisfied if these are equal. Hence, if the problem can be solved by this method,

$$q - q_1 = q_2 \tag{a}$$

The intensity parallel to CD is, on the left,

$$(q + q_1)OP/4\pi c_1 x^3$$

and on the right,

$$q_2 OP/4\pi c_2 x^3$$

Hence, to satisfy (1), we must have

$$(q + q_1)/c_1 = q_2/c_2 \tag{b}$$

Equations (a) and (b) are both satisfied by the values

$$\left.\begin{aligned} q_1 &= -q(c_2 - c_1)/(c_2 + c_1) \\ \text{and} \qquad q_2 &= 2qc_2/(c_2 + c_1) \end{aligned}\right\} \tag{34}$$

so that the above assumptions are justified and the problem solved.

When $c_2/c_1 = \frac{5}{3}$, $q_1 = -q/4$ and $q_2 = 5q/4$. When $c_2/c_1 = \frac{3}{5}$, $q_1 = q/4$ and $q_2 = 3q/8$. The plane diagrams of the field for these two cases are easily constructed from Figs. 23 and 24, or

from the above charges directly, by Maxwell's method, § 14, II. The diagrams of the tubes of displacement for these two cases are given in the upper and lower halves, respectively, of Fig. 59. The dotted lines on the right of the vertical line and the full

$c_1 > c_2$

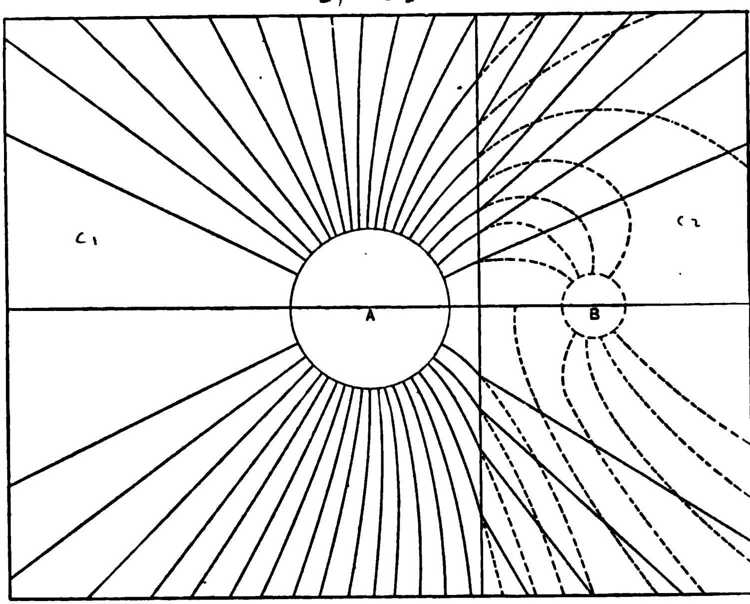


Fig. 59. $c_1 < c_1$

lines on the left are the lines of displacement of Figs. 23 and 24.

The force upon the charged body at A, or the force between the charged body at A and the dielectric 2, is

$$F = qq_1/4\pi c_1(2d)^2 = -q^2(c_2 - c_1)/16\pi d^2 c_1(c_2 + c_1) \qquad (35)$$

If c_2 is greater than c_1, the force is one of attraction, the tubes being concentrated on the side of A toward medium 2; but if c_2 is less than c_1, the force is one of repulsion, the tubes being now concentrated on the opposite side. When $c_2 =$ infinity (35) reduces to (41), II. In the first case the apparent charge upon the interface is negative, in the second positive, and in the last the charge is real and negative, q being supposed positive.

The total force upon the interface can be obtained also by the method of § 6.

Thus the apparent surface density at P is

$$\sigma' = -qd(c_2 - c_1)/2\pi(c_2 + c_1)x^3 \tag{36}$$

The apparent charge dq' upon a zone in the interface of radius OP and width $d(OP)$ is

$$dq' = \sigma' 2\pi OP \cdot d(OP) = \sigma' 2\pi x\,dx = -qd(c_2 - c_1)dx/(c_2 + c_1)x^2$$

The normal intensity at the zone due to the charge q is

$$qd/4\pi c_1 x^3$$

Hence the total force between the interface and the charged body at A is

$$F = -q^2 d^2(c_2 - c_1)/4\pi c_1(c_2 + c_1)\int_d^\infty dx/x^5$$
$$= -q^2(c_2 - c_1)/16\pi c_1 d^2(c_2 + c_1)$$

as in (35).

The total apparent charge upon the interface is

$$\begin{aligned} q' = \int_d^\infty \sigma' 2\pi x\,dx &= -qd(c_2 - c_1)/(c_2 + c_1)\int_d^\infty dx/x^2 \\ &= -q(c_2 - c_1)/(c_2 + c_1) = q_1 \ . \end{aligned} \tag{37}$$

12. Dielectric Sphere in a Uniform Field of Different Permittivity. Let a sphere of permittivity c_2 be introduced into an infinite medium of permittivity c_1 supporting (before the introduction of the sphere) a uniform electric displacement D.

If $c_2 = c_1$, the tubes will remain everywhere unaltered.

If c_2 is greater than c_1, the tubes will bend, crowding into the sphere, thus making D_2 greater than D, until the condition expressed in (3) is satisfied. (If D_2 were to remain equal to D, the lateral pressure $\frac{1}{2}D_2^2/c_2$ across the tubes within the sphere would be less than $\frac{1}{2}D^2/c_1$, the lateral pressure without, and equilibrium could not exist. Also, the voltage between two equipotentials

would be greater along a line not traversing the sphere than along a line passing through the sphere.)

If c_2 is less than c_1, D_2 is less than D, tubes crowding out of the sphere until (3) is satisfied.

We proceed to the exact determination of the electric field within and without the sphere. In accordance with § 48, I., there is but a single field satisfying the conditions of the problem.

With respect to the field within the sphere, we shall make the simplest possible rational assumption, viz., that the displacement D_2 is uniform and in the same direction as the original external displacement D. We shall further assume that the effect of the

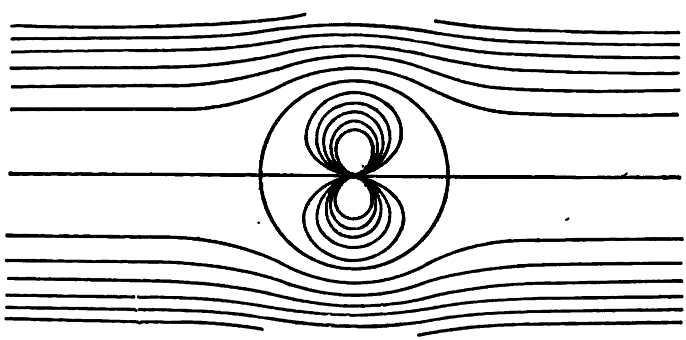


Fig. 60.

sphere on the (originally) uniform field is the same, in the region outside it, as that of a doublet of moment M placed in the original dielectric at the point occupied by the center of the sphere with its axis parallel to D. The probable correctness of this assumption follows from the fact that conductors and dielectrics produce on static fields into which they are introduced effects differing only in degree; and the fact that the effect of a conducting sphere on a uniform field can be represented, in the region outside the sphere, by a doublet at its center.

An attempt at a solution based on these assumptions will obviously satisfy all the electrical conditions except (1) and (2). If in addition D_2 and M can be so chosen as to satisfy these conditions also, the problem will be solved.

To see whether the assumptions made above will satisfy (1) and (2), the radial components of the resultant internal and external displacements and the tangential components of the resultant internal and external intensities at the surface of the sphere must be determined. The radial and tangential displacements at any point, outside the sphere, whose coördinates are R and θ, in the notation of §§ 25 and 26, II., will first be found.

For the radial component due to the doublet (94), II., gives $2M \cos \theta/4\pi R^3$, to which must be added the component $D \cos \theta$ due to the uniform field. For the tangential component due to the doublet (95), II., gives $M \sin \theta/4\pi R^3$, to which $-D \sin \theta$, due to the uniform field, must be added. For the total radial displacement outside the sphere we have, therefore,

$$D_r = 2M \cos \theta/4\pi R^3 + D \cos \theta \tag{38}$$

and for the total tangential displacement,

$$D_t = M \sin \theta/4\pi R^3 - D \sin \theta \tag{39}$$

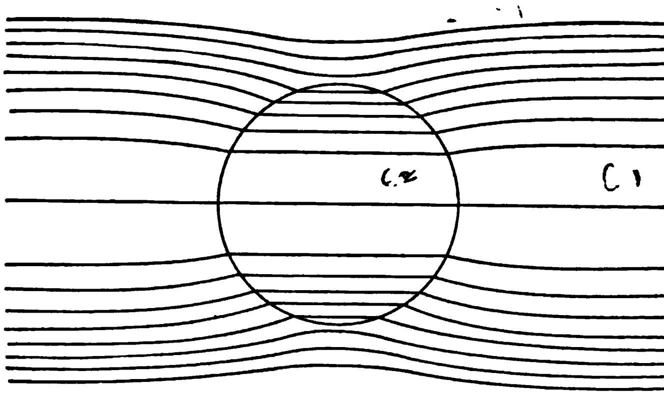


Fig. 61.

At any point inside the sphere, with coördinates R and θ, the radial and tangential components of the displacement are $D_2 \cos \theta$ and $-D_2 \sin \theta$, respectively, independently of the value of R (less than the radius of the sphere).

Let the radius of the sphere be denoted by a. Then, to satisfy (1) and (2), we must have, when $R = a$,

$$2M \cos \theta / 4\pi a^3 + D \cos \theta = D_2 \cos \theta, \quad \text{or} \quad M/2\pi a^3 + D = D_2$$

and $$M \sin \theta / c_1 4\pi a^3 - D/c_1 \cdot \sin \theta = - D_2 \sin \theta / c_2,$$

or $$M/c_1 4\pi a^3 - D/c_1 = - D_2/c_2$$

The solution of these equations gives

$$M = 4\pi a^3 (c_2 - c_1) D/(c_2 + 2c_1) = 4\pi a^3 (c_2 - c_1) D_2 / 3c_2 \qquad (40)$$

and $$D_2 = 3c_2 D/(c_2 + 2c_1)$$

whence $$E_2 = 3c_1 E/(c_2 + 2c_1) \qquad (41)$$

The assumptions made above are therefore justified, and the problem is solved. The uniform field within the sphere is given by (41), and the external field by (38) and (39) on substituting for M its value from (40). This substitution gives

$$D_r = [2a^3(c_2 - c_1)/R^3(c_2 + 2c_1) + 1] D \cos \theta \qquad (42)$$

which becomes, when $R = a$,

$$D_{ra} = 3c_2 D \cos \theta / (c_2 + 2c_1) \qquad (43)$$

also $$D_t = [a^3(c_2 - c_1)/R^3(c_2 + 2c_1) - 1] D \sin \theta \qquad (44)$$

which becomes, when $R = a$,

$$D_{ta} = - 3c_1 D \sin \theta / (c_2 + 2c_1) \qquad (45)$$

When c_2 is greater than c_1, D_2 is greater than D, E_2 is less than E, and M is positive. That is, the doublet is turned with its positive end in the direction of the field. When c_2 is less than c_1, D_2 is less than D, E_2 is greater than E, and M is negative, or the doublet is turned so as to oppose the field. When $c_2 =$ infinity, (41) reduces to $D_2 = 3D$, and $E_2 = 0$; and (38), (39), and (40) to the equations of §§ 25–27, II.

The plane diagrams of the tubes of displacement, drawn by the method of § 14, II., for $c_2/c_1 = 0$, 3, and infinity, respectively, are

given in Figs. 60 (the lines inside the circle, Fig. 32, formed by superposing the uniform field on the field of the doublet, being here annulled), 61, and 62 (from Webser's *Theory of Electricity and Magnetism*, § 194).

The infinite plane passing through the equator of the sphere is an equipotential surface (at zero potential). Hence if this surface is made conducting we shall have on each side half the field just considered terminated by this conducting sheet. Thus we have solved the problem of finding the field terminated by an

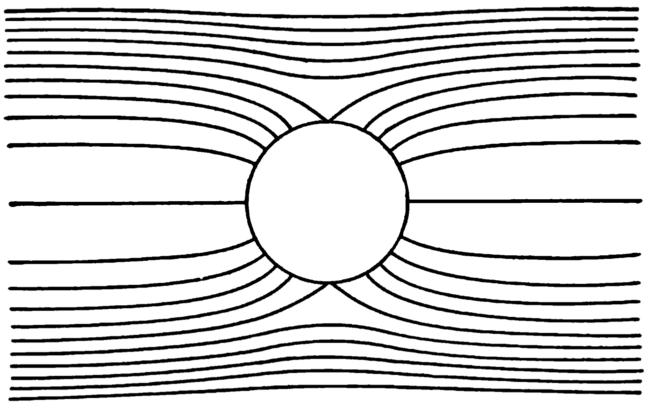


Fig. 62

infinite plane conducting surface with a hemispherical boss upon it of permittivity c_2 differing from that of the dielectric occupying the rest of the field (c_1).

The electric surface density at any point of this plane distant R from the center of the hemisphere is

$$\sigma = \pm D_2 = \pm 3c_2D/(c_2 + 2c_1) \tag{46}$$

when R is less than a, and

$$\sigma = \pm D_t(\theta = 90°) = \pm [a^3(c_2 - c_1)/R^3(c_2 + 2c_1) - 1]D \tag{47}$$

when R is greater than a.

The intensity of electrisation of the sphere is uniform and equal to

$$J = D_2[(c_2 - c_1)/c_2] = 3D(c_2 - c_1)/(c_2 + 2c_1) \tag{48}$$

(40) may now be written, in terms of J,

$$M = \tfrac{4}{3}\pi a^3 J \tag{49}$$

so that the intensity of electrisation of the sphere might be defined as its *electric moment per unit volume*, the electric moment denoting the moment of the doublet producing the same effect on the external field as that of the sphere.

The difference between the actual intensity E_2 in the sphere and the original intensity E of the uniform field is called the *self-deëlectrising force* or *intensity* in the sphere due to its poles or apparent charges, and will be denoted by E'. Thus

$$\begin{aligned} E' &= E_2 - E = -D_2(c_2 - c_1)/3c_1c_2 \\ &= -E(c_2 - c_1)/(c_2 + 2c_1) = -J/3c_1 \end{aligned} \tag{50}$$

The apparent electric surface density at a point whose coördinates are a and θ is

$$\begin{aligned} \sigma' &= D_2[(c_2 - c_1)/c_2] \cos\theta = J\cos\theta \\ &= 3[(c_2 - c_1)/(c_2 + 2c_1)]D\cos\theta \end{aligned} \tag{51}$$

The total apparent charge upon one half of the sphere between a pole and the equator is

$$q' = \pm\pi a^2 J = \pm\pi a^2[(c_2 - c_1)/c_2]D_2 \tag{52}$$

13. Infinite Dielectric or Conducting Cylinder in a Uniform Field. Making use of § 20, II., we can obtain, by the method of the preceding article, the electric field in and about an infinite circular cylindrical dielectric of permittivity c_2 immersed in the uniform field of an infinite medium whose permittivity is c_1. For, as will be seen, it is possible to satisfy all the conditions by assuming the displacement external to the cylinder to be the resultant of the original uniform displacement and the displacement of a line doublet of moment M, suitably chosen, at its axis, all in the original dielectric, and the internal displacement to be uniform and parallel to the original displacement. If the radius of the cylinder is a, and if D and D_2 denote the original uniform

displacement and the actual displacement within the cylinder, respectively, the conditions (1) and (2) to be satisfied at the interface are obviously

$$D\cos\theta + M\cos\theta/2\pi a^2 = D_2\cos\theta$$

and

$$-D\sin\theta/c_1 + M\sin\theta/c_1 2\pi a^2 = -D_2\sin\theta/c_2$$

from which we obtain

$$D_2 = 2c_2 D/(c_2 + c_1) \tag{53}$$

and $M = \pi a^2[(c_2 - c_1)/c_2]D_2 = 2\pi a^2[(c_2 - c_1)/(c_2 + c_1)]D$ (54)

Hence outside the cylinder, at a distance R from its axis, the radial component of the total displacement is

$$\begin{aligned} D_r &= M\cos\theta/2\pi R^2 + D\cos\theta \\ &= [a^2/R^2\cdot(c_2 - c_1)/(c_2 + c_1) + 1]D\cos\theta \end{aligned} \tag{55}$$

and the tangential component is

$$D_t = [a^2/R^2\cdot(c_2 - c_1)/(c_2 + c_1) - 1]D\sin\theta \tag{56}$$

while within the cylinder the displacement is uniform and given by (53).

The apparent electric surface density is

$$\sigma' = [(c_2 - c_1)/c_2]D_2\cos\theta = J\cos\theta \tag{57}$$

The total positive or negative apparent charge on half of unit length of the cylinder is

$$q_1 = J \times 2a \times 1 = 2aJ \tag{58}$$

The self-deëlectrising force of the apparent charges is

$$\begin{aligned} E' &= D_2/c_2 - D/c_1 = -(c_2 - c_1)/2c_1c_2\cdot D_2 \\ &= -E(c_2 - c_1)/(c_2 + c_1) = -J/2c_1 \end{aligned} \tag{59}$$

14. Dielectric Spherical Shell in a Uniform Field. Electric Screen. If instead of the solid sphere of § 12, we have a spherical shell of permittivity c_2, with inner and outer radii b and a respec-

tively, surrounding and surrounded by a medium of permittivity c_1 supporting an (originally) uniform field with displacement D, we can find the field by an extension of the method used in the two preceding articles.

Guided by the results obtained for the solid sphere, we shall assume (1) that within the inner surface of the shell the displacement, D_3 is uniform and parallel to that of the original field; (2) that within the shell the displacement is the vector sum of a uniform displacement D_2 parallel to D and the displacement due to a point doublet of moment M_b at the center of the spheres; and (3) that the displacement outside the shell is the vector sum of the uniform displacement D, the displacement due to the doublet of moment M_b, and the displacement due to a second doublet of moment M_a, also placed at the center of the spheres, the axes of both doublets being parallel to D. It will now be shown that these assumptions satisfy (1) and (2).

At the outer interface the conditions to be satisfied by the normal displacement and tangential intensity are, respectively,

$$(D + 2M_a/4\pi a^3 + 2M_b/4\pi a^3) \cos\theta = (D_2 + 2M_b/4\pi a^3) \cos\theta$$

and

$$(-D/c_1 + M_a/c_1 4\pi a^3 + M_b/c_1 4\pi a^3) \sin\theta$$
$$= (-D_2/c_2 + M_b/c_2 4\pi a^3) \sin\theta$$

At the inner interface the conditions are

$$(D_2 + 2M_b/4\pi b^3) \cos\theta = D_3 \cos\theta$$

and $$(-D_2/c_2 + M_b/c_2 4\pi b^3) \sin\theta = -D_3/c_1 . \sin\theta$$

Cos θ and sin θ will divide out, and the equations are satisfied by the following values of the assumed moments and displacements:

$$D_3 = 9c_1c_2D/[9c_1c_2 + 2(c_2 - c_1)^2(1 - b^3/a^3)] \qquad (60)$$

$$D_2 = [(2c_2 + c_1)/3c_1]D_3 \qquad (61)$$

$$M_a = 2\pi a^3(D_2 - D) \qquad (62)$$

and $$M_b = -4\pi b^3/3 \cdot [(c_2 - c_1)/c_1]D_3 \qquad (63)$$

The relation betwen D_3/D and b/a is given in the accompanying table (Table I.) for the cases in which $c_2/c_1 = 100$ and 1000.

These excessive values of c_2/c_1 do not occur in electrostatics, but are assumed here for the sake of the much more important magnetic analogue, § 23, XI. The greater the ratio of c_2 to c_1 and the smaller the ratio of b to a, the less is D_3 in comparison with D, that is, the shell forms a more effective *screen* from electrical influences for the region within it. When $c_2/c_1 =$ infinity, that is, when the shell is conducting, $D_3/D = 0$ for all values of b/a, and the shell is a perfect electrical screen (when the field is static).

15. Dielectric Cylindrical Shell in a Uniform Field. Electric Screen. The field of an infinitely long circular cylindrical shell,

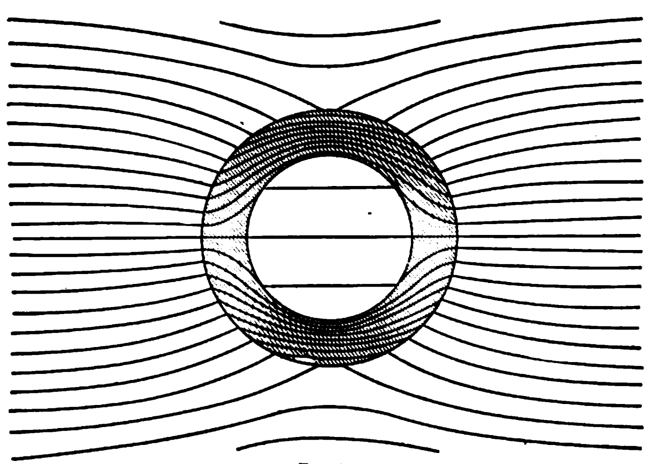


Fig. 63.

of permittivity c_2 and with inner and outer radii b and a, when immersed in an infinite medium of permittivity c_1 supporting an (originally) uniform displacement D can be obtained in exactly the same way, by making use of line doublets (§ 20, II.) instead of point doublets.

Let D, D_2, D_3, M_a, and M_b have the same meanings as in § 14, except that cylinder and cylindrical must be substituted for sphere and spherical and line doublet for point doublet. Then we have, to satisfy (1) and (2), at the outer interface,

$$(D + M_a/2\pi a^2 + M_b/2\pi a^2)\cos\theta = (D_2 + M_b/2\pi a^2)\cos\theta$$

and

$$(-D/c_1 + M_a/c_1 2\pi a^2 + M_b/c_1 2\pi a^2)\sin\theta = (-D_2/c_2 + M_b/c_2 2\pi a^2)\sin\theta$$

and at the inner interface,

$$(D_2 + M_a/2\pi b^2)\cos\theta = D_3\cos\theta$$

and $$(-D_2/c_2 + M_b/c_2 2\pi b^2)\sin\theta = -D_3/c_1 \cdot \sin\theta$$

Solving these equations, we obtain, as the only solution of the problem,

$$D_3 = 4Dc_1c_2/\left[(c_2+c_1)^2 - (b^2/a^2)(c_2-c_1)^2\right] \qquad (64)$$

$$D_2 = (c_2+c_1)/2c_1 \cdot D_3 \qquad (65)$$

$$M_a = 2\pi a^2(D_2 - D) \qquad (66)$$

and $$M_b = -\pi b^2\left[(c_2-c_1)/c_1\right]D_3 \qquad (67)$$

The relation between D_3/D and b/a is given in the accompanying table (Table I.) for the cases in which $c_2/c_1 = 100$ and 1000. When $c_2/c_1 =$ infinity, or when the shell is a conductor, $D_3/D = 0$ for all values of b/a. The cylindrical shell, like the spherical shell of § 14, forms an *electrical screen*, the remarks at the close of § 14 applying equally well to both forms.

The plane diagram of the tubes of displacement when $c_2/c_1 = 10$ and $b/a = \frac{2}{3}$ is given in Fig. 63 (from Webster's *Theory of Electricity and Magnetism*, § 198).

TABLE I.

SCREENING EFFECT OF SPHERICAL* AND CYLINDRICAL DIELECTRIC SHELLS.

b/a	D_3/D when $c_2/c_1 = 100$.		D_3/D when $c_2/c_1 = 1000$.	
	Spher. Shell.	Cyl. Shell.	Spher. Shell.	Cyl. Shell.
0.0	$\frac{1}{23}$	$\frac{1}{26}$	$\frac{1}{223}$	$\frac{1}{251}$
0.1	$\frac{1}{23}$	$\frac{1}{25}$	$\frac{1}{223}$	$\frac{1}{248}$
0.2	$\frac{1}{23}$	$\frac{1}{25}$	$\frac{1}{221}$	$\frac{1}{241}$
0.3	$\frac{1}{22}$	$\frac{1}{23}$	$\frac{1}{216}$	$\frac{1}{228}$
0.4	$\frac{1}{22}$	$\frac{1}{22}$	$\frac{1}{209}$	$\frac{1}{211}$
0.5	$\frac{1}{20}$	$\frac{1}{19}$	$\frac{1}{195}$	$\frac{1}{188}$
0.6	$\frac{1}{18}$	$\frac{1}{17}$	$\frac{1}{175}$	$\frac{1}{161}$
0.7	$\frac{1}{15}$	$\frac{1}{14}$	$\frac{1}{146}$	$\frac{1}{128}$
0.8	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{109}$	$\frac{1}{91}$
0.9	$\frac{1}{7}$	$\frac{2}{11}$	$\frac{1}{67}$	$\frac{1}{48}$
0.99	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{23}$	$\frac{1}{8}$
1.0	1.0	1.0	1.0	1.0

* The data for the spherical shell are taken from J. J. Thomson's *Elements of the Mathematical Theory of Electricity and Magnetism*, § 161.

CHAPTER V.

REVERSIBLE THERMAL EFFECT DURING ELECTRISATION. ELECTROSTRICTION.

1. Reversible Thermal Effect During Electrisation. Let a condenser be carried through a reversible cyclic process as follows, the external pressure upon the dielectric (*e. g.*, the atmospheric pressure) being kept constant:

(1) The voltage V being kept constant, let the condenser be heated from the absolute temperature t to the absolute temperature $t + dt$. If ρ, s, and τ denote the density, specific heat, and volume, respectively, of the dielectric, the heat absorbed by the condenser during this process is $H = \rho s \tau dt$.

(2) At the temperature $t + dt$, at which the capacity of the condenser is $S + dS/dt\, dt$, let the voltage be increased by dV. The energy of the condenser increases by $\frac{1}{2}(S + dS/dt\, dt)d(V^2) = (S + dS/dt\, dt)VdV$.

(3) Let the condenser be cooled to the original temperature t while the voltage remains constant $(V + dV)$.

(4) Let the voltage be reduced to its original value V, the condenser thus losing an amount of energy equal to

$$\tfrac{1}{2}Sd(V^2) = SVdV$$

The condenser is now in its original condition.

The total work done upon the condenser (exclusive of work done in heating) during the complete cycle is

$$dW = (S + dS/dt\, dt)VdV - SVdV = VdS/dt\, dt \cdot dV$$

The quantity of heat given to the condenser (exclusive of that given out) is

$$H = \rho s \tau\, dt$$

Hence, by the second law of thermodynamics, viz., $H/t = -dW/dt$, we have

$$\rho s\tau\, dt/t = -VdVdS/dt$$

or

$$dt/dV = -tV/\rho s\tau \cdot dS/dt \qquad (1)$$

If therefore we assume that dS/dt, s, etc., are independent of V, which is certainly near the truth, the total reversible temperature change when the condenser is charged from $V = 0$ to $V = V$ is

$$\Delta t = -t/\rho s\tau \cdot dS/dt \int_0^V VdV = -t/\rho s\tau \cdot 1/S \cdot dS/dt \cdot \tfrac{1}{2}SV^2 \quad (2)$$

If the dielectric is homogeneous and isotropic, we have

$$1/S \cdot dS/dt = 1/cL \cdot d(cL)/dt = 1/c \cdot dc/dt + 1/L \cdot dL/dt = k_t + a \quad (3)$$

where L is the length of any line drawn in the dielectric (S and dS being proportional by the same factor to the product of the permittivity and such a length), and k_t and a are written for $1/c \cdot dc/dt$, the coefficient of increase of c with temperature, and $1/L \cdot dL/dt$, the coefficient of linear expansion with temperature, respectively. By substituting (3) in (2), we obtain

$$\Delta t = -t/\rho s\tau \cdot (k_t + a)\tfrac{1}{2}SV^2$$

or

$$\Delta t/t = -(k_t + a)/\rho s\tau \cdot \tfrac{1}{2}SV^2 \qquad (4)$$

If the field of the condenser is uniform, (4) becomes

$$\Delta t/t = -(k_t + a)/\rho s \cdot \tfrac{1}{2}cE^2 = -(k_t + a)\,U/\rho s \qquad (5)$$

Since, moreover, any electric field is uniform in its infinitesimal parts, (5) is perfectly general.

In all the above the effects of conduction, radiation, etc., are neglected, and no intrinsic electrisation (VI.) is supposed to be present.

For all solids yet investigated $(k_t + a)$ is positive. Hence a condenser with such a dielectric is cooled by charging and heated by discharging. For nearly all liquids $(k_t + a)$ is negative. According to experiments by W. Cassie (*Phil. Trans.*, A, 1890) k_t

(or $k_t + a$) at 50° C. is about -0.006 for glycerine and $+0.0004$ for mica. For recent literature see *Ann. der Physik*, Vol. 10, p. 748, 1903.

[Analogous magnetic quantities being substituted in (5) for the electric quantities there occurring, the equation is valid for the magnetic case (see § 23, XI.).]

2. Electrostriction. Change in Volume of Dielectric when Electrised. When a condenser is charged at constant temperature its dielectric, or dielectrics, would be expected, in general, to suffer changes in volume and changes in linear dimensions. These phenomena, as yet largely hypothetical, are included under the general head of *electrostriction*. The alterations in volume, etc., can be deduced from the principles already developed, in connection with the principle of the conservation of energy.

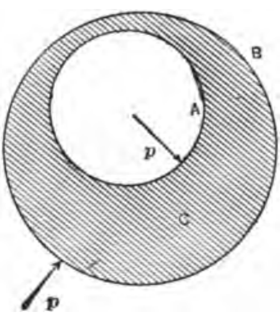


Fig. 64.

In all that follows it will be assumed that the condenser plates are always in contact with the dielectric and that they follow accurately without appreciable elastic reaction the motion of its surfaces, as, for example, coats of gold leaf or tin foil. Complete absence of intrinsic displacement will also be assumed.

First we shall find the change in volume. Consider a condenser ABC, Fig. 64, whose dielectric C occupies the volume τ and possesses the permittance S when charged to the voltage V and subjected to the uniform pressure p (which may have any value, including 0) over its surfaces.

(1) While V is kept constant, let the volume be increased by $d\tau$. The energy of the condenser will increase by $-p\,d\tau$. The increase of volume will, in general, be accompanied by an increase dS in the permittance. Now let the volume $(\tau + d\tau)$ be kept constant while the voltage is increased by dV. The energy increases by $\frac{1}{2}(S + dS)d(V^2) = (S + dS)VdV$. The total increase in energy during the process is

$$dW_1 = -pd\tau + (S + dS)VdV$$

(2) Let us start with the condenser in the same condition as at the beginning of (1) and bring it to the same final state by a slightly different process. While the volume remains constant (τ), let the voltage be increased by dV. This will increase the energy of the condenser by $SVdV$, and, in general, the pressure by an amount dp. Now let the voltage $(V + dV)$ be kept constant while the volume is increased by $d\tau$. The energy will increase by $-(p + dp)d\tau$. The total increase in the condenser's energy is thus

$$dW_2 = SVdV - (p + dp)\,d\tau$$

By the principle of the conservation of energy, $dW_1 = dW_2$. Hence

$$VdS\,dV = -\,dp\,d\tau$$

or

$$d\tau/dV = -\,VdS/dp \tag{6}$$

For ordinary charges dS/dp will be sensibly independent of V. Hence we have for the total change in τ when the condenser is charged from a neutral state to the voltage V,

$$\Delta\tau = -\,dS/dp\int_0^V VdV = \tfrac{1}{2}\,SV^2(-\,1/S\cdot dS/dp) \tag{7}$$

Homogeneous Isotropic Dielectric. If the dielectric is homogeneous and isotropic, (7) may be simplified. For in this case S and dS are proportional by the same factor to the product of the permittivity and the linear dimensions of the condenser.

Hence if we denote by L the length of any line drawn in the dielectric, we have

$$-1/S\cdot dS/dp = -1/cL\cdot d(cL)/dp$$
$$= -1/c\cdot dc/dp - 1/L\cdot dL/dp = (k_p + b/3)$$

where $b =$ the coefficient of compressibility of the dielectric $= -3/L\cdot dL/dp$, and $k_p = -1/c\cdot dc/dp =$ the coefficient of diminution with pressure, or increase with traction, of the permittivity c. Thus (7) becomes

$$\Delta\tau = \tfrac{1}{2}SV^2(k_p + b/3) \tag{8}$$

In order that (7) or (8) may hold when the dielectric is a fluid, the dielectric must be completely surrounded by the con-

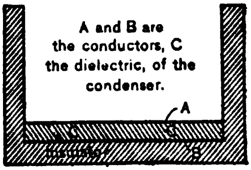


Fig. 65.

denser plates, or the plates must be so arranged that they are kept apart by the pressure of the fluid only, and sensibly all the tubes must be contained in the fluid, as in Fig. 65.

If the field of the condenser is uniform, $\tfrac{1}{2}SV^2 = \tau\,\tfrac{1}{2}cE^2$, and (8) may be written, on division by τ,

$$\Delta\tau/\tau = \tfrac{1}{2}cE^2(k_p + b/3) \tag{9}$$

3. Change in Length of a Line Normal to a Uniform Electric Field in a Solid Isotropic Homogeneous Dielectric During Electrisation. Let e denote the thickness of the dielectric of a parallel plate condenser (the plates always remaining in contact with the single dielectric), and L and L' the lengths of the edges normal to e of a rectangular prism of the dielectric. We shall find the change in L when the condenser is charged to the voltage V. The dielectric in all that follows will be supposed homogeneous and isotropic.

(1) The voltage V being kept constant, let the dielectric be subjected to a traction in the direction of L, the stress across the area $L'e$, normal to L, being Q_1. This will increase the energy of the prism by $Q_1 dL$, and will, in general, alter its capacity by an amount dS. Now let the length $L + dL$ remain constant while the voltage is increased by dV. This will increase the energy by $(S + dS) \cdot VdV$. The total increase in the energy is

$$dW_1 = Q_1 dL + (S + dS) VdV$$

(2) Let the condenser be brought from the same initial state to the same final state as before by a different process. First let the voltage increase by dV, while L remains constant, which will increase the energy by $SVdV$ and, in general, the traction by dQ_1. Then, the voltage $(V + dV)$ being kept constant, let L be increased by dL, which will increase the energy by $(Q_1 + dQ_1)dL$. The total increase is

$$dW_2 = SVdV + (Q_1 + dQ_1)dL$$

As in § 2, $dW_2 = dW_1$, hence

$$dL/dV = VdS/dQ_1 \qquad (10)$$

Since, for small changes at least, dS/dQ_1 must be sensibly independent of V, (10) gives for the total change in L when the condenser is charged from $V = 0$ to $V = V$,

$$\Delta L = dS/dQ_1 \int_0^V VdV = \tfrac{1}{2} SV^2 (1/S \cdot dS/dQ_1) \qquad (11)$$

Since $S = cLL'/e$,

$$1/S \cdot dS/dQ_1 = 1/c \cdot dc/dQ_1 + 1/L \cdot dL/dQ_1 + 1/L' \cdot dL'/dQ_1 - 1/e \cdot de/dQ_1$$

Moreover, $1/L' \cdot dL'/dQ_1 = 1/e \cdot de/dQ_1$. Hence we have, putting $dQ_1 = L'e \cdot dq_1$, simplifying, and dividing by L,

$$\Delta L/L = \tfrac{1}{2} cV^2/e^2 \cdot (1/c \cdot dc/dq_1 + 1/L \cdot dL/dq_1) \qquad (12)$$

Now $1/L \cdot dL/dq_1$ is the reciprocal of the stretch modulus, and will be denoted by M. Also, $1/c \cdot dc/dq_1$ is the coefficient of in-

crease of the permittivity with traction normal to the lines of displacement, and will be denoted by k_1. Thus (12) becomes

$$\Delta L / L = \tfrac{1}{2} c V^2 / e^2 \cdot (M + k_1) = \tfrac{1}{2} c E^2 (M + k_1) \tag{13}$$

The above results, deduced for the uniform field of a parallel plate condenser, will hold good, without sensible error, for a condenser of any form, such as a cylindrical or spherical condenser, in which the conductors are parallel and so close together that E is sensibly of the same magnitude throughout the dielectric.

4. Change in Length of a Line in the Direction of a Uniform Electric Field in a Solid Isotropic Homogeneous Dielectric. Making use of the parallel plate condenser of the last article, and of the same general method, but applying a traction $Q_2 = LL'q_2$ parallel to the lines of intensity, we obtain

$$de / dV = V dS / dQ_2 \tag{14}$$

$$\Delta e = \tfrac{1}{2} S V^2 (1 / S \cdot dS / dQ_2) \tag{15}$$

and

$$\begin{aligned} \Delta e / e &= -\tfrac{1}{2} c V^2 / e^2 \cdot [M(2r + 1) - k_2] \\ &= -\tfrac{1}{2} c E^2 [M(2r + 1) - k_2] \end{aligned} \tag{16}$$

where r denotes Poisson's ratio, and $k_2 = 1 / c \cdot dc / dq_2$ is the coefficient of increase of c with traction parallel to the lines of intensity.

The results just established hold good, like those of § 3, for a thin condenser with parallel plates of any form.

It is easy to see that

$$k_p = 2k_1 + k_2 \tag{17}$$

5. Theory and Experiments. A rigorous treatment of the general theory of electrostriction, together with a résumé of most of the experimental and theoretical investigations upon the subject, is contained in a recent memoir by P. Sacerdote (*Ann. de Chim. et de Phys.* (7), 20, p. 289, 1900). Satisfactory experi-

ments upon the values of the coefficients k_p, k_1, and k_2, as well as entirely satisfactory experiments upon the quantities $\Delta\tau/\tau$, $\Delta e/e$ and $\Delta L/L$, have not yet been performed (See the *Philosophical Magazine* and *Nuovo Cimento*, 1900–1902, for some of the most recent and best results).

§§ 2–4 are based largely upon portions of the above-mentioned memoir by Sacerdote, with simplifications.

CHAPTER VI.

ELECTRIC ABSORPTION. ELECTRETS.

1. Electric Absorption. In all that precedes electric displacement has been treated as a perfectly elastic phenomenon; that is, the relation $D = cE$ (analogous to Hooke's law) has been assumed to hold universally with c at every point a constant, independent of the time. On this assumption, the capacity of a condenser, which is proportional to c, would be invariable with the time of charging. This appears from experiment to be accurately true for dielectrics whose homogeneity is perfect, for example, gases, pure paraffine, and pure calc spar; but it is by no means true in general, as the experiments described below demonstrate.

Let a condenser whose dielectric is not homogeneous, with its plates connected to the quadrants of an electrometer, be charged to a given potential difference and then insulated from the battery. The potential difference will gradually diminish, approaching a limit sometimes considerably below its initial value. If now the condenser is short-circuited, the potential difference becomes zero; but it gradually reappears, unchanged in sign but much smaller in magnitude, after the condenser is again insulated. The remnant of the original charge, whose presence is proved by the existence of this potential difference, is called a *residual charge*. If the operation of short-circuiting and insulating is repeated, the same phenomena recur, the potential difference developed after insulation being smaller each time and finally becoming insensible. The disappearance of the phenomena is of course hastened by the "leakage" of the condenser, if appreciable, arising from the conductivity of its dielectric, from the presence of moisture, etc.

If the condenser is charged for a long time in one direction, then for a much shorter time with the poles of the charging battery reversed, and then short-circuited and insulated, a potential difference similar to that last applied will first appear, reach a maximum, diminish to zero, change sign and continue to increase in the direction of the potential difference first applied.

Or the following equivalent phenomena may be observed. On the condenser's being connected with a constant battery, its charge usually reaches very quickly almost its final value, but the charge goes on gradually increasing, sometimes considerably exceeding its initial magnitude. On short-circuiting the condenser most of the charge disappears; but after insulation for a short time a second discharge in the same direction may be obtained, and so on, till the discharges become too small to be perceptible.

Also, if the condenser is charged for a long time in one direction, then for a much shorter time in the opposite direction, and then short-circuited and insulated, a residual charge (and corresponding discharges, if the condenser is repeatedly short-circuited) similar in sign to the last charge will at first appear, but will be succeeded by a residual charge similar in sign to that of the charge first applied.

The appearance of the residual charge in all the above described experiments is hastened by subjecting the condenser to mechanical shocks.

Two possible explanations of the phenomena of *electric absorption*, as the phenomena just described are called on account of what was once regarded as the *soaking in* of the electric charge with the time, have been given.

(1) The general analogy between electric strain and stress and mechanical strain and stress, together with the fact that absorption does not occur in free æther or in gases, whose elasticity is perfect, and is very marked in a substance like glass, whose elasticity is extremely imperfect, has led to the suggestion that electric absorption is due to the imperfect electric elasticity of the

dielectrics which exhibit it. The phenomena of electric absorption are exactly analogous to the phenomena of elastic after-action. Thus, if the spring of § 36, I., is not perfectly elastic (and no solid body is perfectly elastic) the elongation (analogous to electric charge) produced by a certain applied force (analogous to e.m.f. of charging battery), equal and opposite to the elastic return-force of the spring (potential difference), will not remain constant with the time, but after reaching almost immediately a value usually very near the final value, will gradually increase. If now the force is removed (condenser short-circuited), the elongation will not become zero at once, but the spring can exert no force by virtue of the remaining elongation (residual charge when potential difference = 0). If now the spring is clamped (condenser insulated), the elongation gradually becomes elastic (residual charge becomes available, potential difference increases from zero), and the spring exerts a force upon the clamp (residual potential difference). If the clamp is removed (short-circuit), the elongation will again suddenly diminish, and so on. Also, if the spring is clamped, extended for a long time, and then compressed for a much shorter time (condenser charged successively in opposite directions) and then released (short-circuit), the residual compression will gradually reach zero, and then become a residual elongation which will diminish much more slowly to zero (condenser's dischargers will be for a short time in one direction, then for a much longer time, until the whole residual charge has disappeared, in the opposite direction).

(2) Maxwell has developed a theory according to which the phenomena of absorption can not occur if the dielectric is perfectly homogeneous throughout, but must occur whenever the ratio of the permittivity to the conductivity is not constant for all parts of the dielectric, even if none of the constituents alone exhibits the phenomena. This conclusion, according to which electric absorption is due to heterogeneity of structure, is supported by experiments of Rowland and Nichols, Muraoka, and others. It is quite possible that deviation from perfect elas-

ticity is closely connected with heterogeneity of structure, and that the two explanations are not independent of one another. Thus glass is extremely heterogeneous, possesses very imperfect elasticity, and shows the phenomena of absorption in a marked manner.

2. Dielectric Absorption Hysteresis. If a condenser whose dielectric is absorbent is rapidly charged to a voltage V', short-circuited, charged in the opposite direction to a voltage $-V'$, short-circuited, charged again to voltage V', and the process repeated a number of times at the same rate (by connecting the condenser to the poles of an alternating current dynamo, for example), it is evident from what precedes and the principle of symmetry that the relation between the charge q and voltage V of the condenser may be represented by a closed symmetrical curve, such as that in Fig. 66. When the voltage has dropped from the

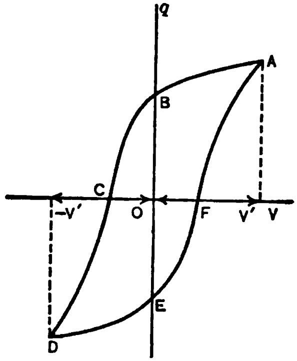


Fig. 66.

value V' at A to O at B, a residual charge OB is left, and disappears entirely only when the voltage reaches the negative value OC. As the voltage increases negatively to $-V'$ at D, the charge increases negatively, and falls to the value OE when the voltage again becomes zero. The residual charge again disappears when $V = OF = -OC$, etc., the charge thus always *lagging* behind the voltage.

If however the condenser is carried very slowly through the cycle of charging, discharging, etc., or if the dielectric is one which does not exhibit electric absorption, the curve is found to reduce to a straight line, and the area of the cycle therefore to zero. (Cf. Beaulard, *Journal de Physique* (3), 9, 422, 1900.)

The phenomena are therefore not analogous to those of magnetic hysteresis (§ 39, XIII.), which are almost wholly independent of the time in which a cycle is completed and are not dependent, except to a very slight extent, upon anything similar to viscosity or absorption. The term *hysteresis* may be used to designate the electrical phenomenon described in this article, on account of the *lagging* effect mentioned, but this term, if so used, should be coupled with the word *viscosity* or *absorption* in order to avoid the incorrect inference that the phenomenon is physically analogous to magnetic hysteresis.

3. Energy Dissipated in Dielectric Hysteresis.—The area of the curved figure $ABCDEFA$, Fig. 66, is

$$H = \int V\,dq \tag{1}$$

for the whole cycle, and thus represents the excess of the electrical work done in charging the condenser (in both directions) over the electrical energy given out when the condenser is discharged (in both directions) (see § 37, I.). This quantity of energy must therefore be transformed into heat during each completion of the cycle.

(1) may be written

$$H = \int\int\int E dL \cdot dS dD = \int\int E dD \cdot d\tau$$

dL and dS being elements of a line of intensity and an equipotential surface, respectively, and $d\tau$ being the element of volume $dLdS$. We have therefore for the energy dissipated per unit volume per cycle at a point in any dielectric where the intensity and displacement are denoted by E and D,

$$dH/d\tau = \int E dD \tag{2}$$

the integration being extended throughout a complete cycle. See § 38, I.

4. Intrinsic Displacement and Intensity, Electrets, etc. A dielectric electrised or retaining its electrisation, like the dielectric of a condenser after absorption has occurred, partially or wholly under the action of internal forces, no external field, and therefore *no* potential difference or field intensity ($E = -dV/dL$) within the dielectric itself (as when the condenser is short-circuited), being necessarily present, is said to possess *intrinsic* electrisation or displacement, and to be under the action of an *intrinsic* electric intensity or force, denoted by e, in the direction of the displacement. A dielectric in this state is called an *electret.*

The intensity $E = -dV/dL$, § 3, is zero at two points of the cycle for which D (the total displacement, *redefined* by Gauss's theorem * as dq/dS at a conducting surface, the theorem being *assumed* to hold for intrinsic as well as for elastic displacement) has finite values, while D (or q) is zero at two points for which E (or V) has finite values.

Thus if we assume the relation $c = D/E$ (by which D was *defined* in the case of elastic displacement) to hold in the case of intrinsic displacement, c, as the cycle is traversed, will pass through all values from $+\infty$ at B to $-\infty$ at E, Fig. 66.

If, however, we introduce the conception of intrinsic intensity e, if we denote the field intensity $-dV/dL$ by E' instead of E at a point where intrinsic displacement exists, and if we denote the vector sum of e and E' by $(e + E') = E$, the *total* or *impressed* intensity, we may define e by the relation

$$D = cE = c(e + E') \tag{4}$$

With this understanding, the relation $D = cE$ holds univer-

* A more rigorous and general definition of D, analogous to the general definition of B, (63), XIII., is obtained from (2), XV. Thus

$$D = \int_0^t curl\, H\, dt \tag{3}$$

the dielectric being in a neutral state at the time $t = 0$.

sally and leads to no impossible values of c, since e and D have always the same direction. In this chapter E' will be used to denote the field intensity $-dV/dL$; but elsewhere, except where the contrary is stated, e will be assumed equal to 0 and E' equal to E.

5. Uniformly Electrised Spherical Electret. Suppose the electrisation of the sphere of § 12, IV., to become partially intrinsic. Then let the external charges "producing" the (originally) uniform field be removed. Then the only remaining "charges" are the fictitious charges upon the surface of the sphere, whose density is given by

$$\sigma' = J \cos \theta \qquad (5)$$

where J denotes the intensity of the remaining electrisation.

Inside the sphere there is an intrinsic intensity e maintaining the displacement and a self-deëlectrising field intensity equal to

$$E' = -J/3c_1 \qquad (6)$$

The external field is the part outside the sphere of the field connected with the doublet of moment

$$M = \tfrac{4}{3} \cdot \pi a^3 J \qquad (7)$$

at the center of the sphere.

Maxwell's plane diagram of the complete field is given in Fig. 67 (from Maxwell's *Treatise*, § 143). If the lines of displacement within the sphere are directed from S to N, the lines of intensity have the direction NS.

The quantities M, E', σ', and J can all be expressed in terms of the internal displacement D_0 of the sphere.

Thus we find from the relations (2) IV., (94) II., and (7), when $\theta = 0$ and $R = a$: $D_0 = M/2\pi a^3$; or

$$M = 2\pi a^3 D_0 = \tfrac{4}{3} \pi a^3 J \qquad (8)$$

The relations (1) IV., (95) II., and (6) give when $\theta = 90°$ and $R = a$,

$$E' = M/4\pi c_1 a^3 = -D_0/2c_1 = -J/3c_1 \qquad (9)$$

The relation $J = D_0 - c_1 E'$ [(7), IV.], or either of the last two equations alone, gives

$$J = D_0 + D_0/2 = \tfrac{3}{2} D_0 \tag{10}$$

From this equation we have

$$\sigma' = J \cos\theta = \tfrac{3}{2} D_0 \cos\theta \tag{11}$$

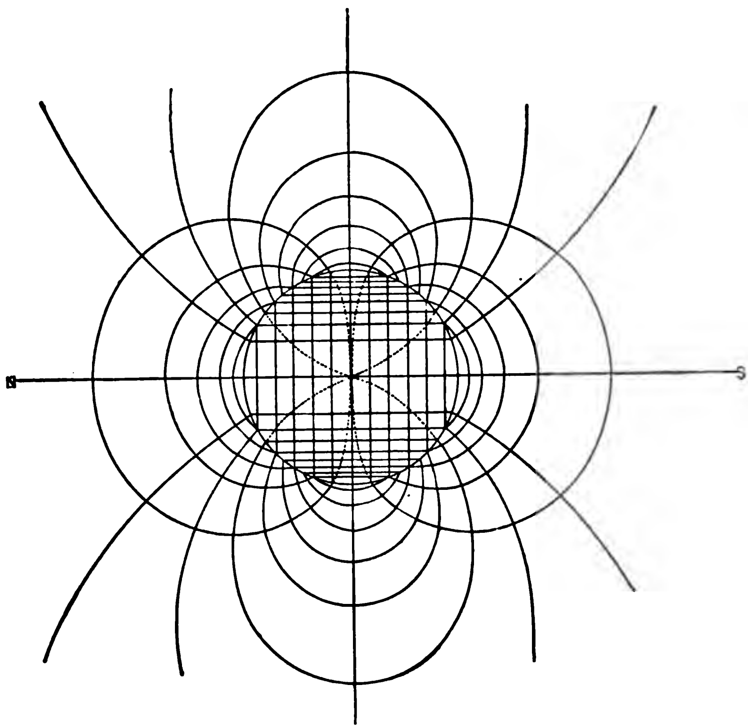


Fig. 67.

The magnitude of the strength of each pole of the sphere (distributed over a hemisphere) is

$$q' = \pi a^2 J = \tfrac{3}{2}\pi a^2 D_0 = \tfrac{3}{2}\,\Pi \text{ (see below)} \tag{12}$$

The total electric flux through the sphere from the negative to the positive pole, and back again outside the sphere from the

positive pole to the negative pole, all the tubes of displacement being *closed*, is

$$\Pi = \pi a^2 D_0 = \tfrac{2}{3}\, \pi a^2 J = \tfrac{2}{3} q' \tag{13}$$

The total flux of the intensity from or to one of the poles is

$$\Pi' = \pi a^2 D_0 / c_1 - \pi a^2 E' = \tfrac{3}{2}\, \Pi / c_1 = q' / c_1 \tag{14}$$

In terms of the intensities and the permittivity c_2 of the sphere, the internal displacement, denoted above by D_0, is

$$D_0 = c_2 (e + E') \tag{15}$$

Since e and E' have opposite directions, D_0 is less than if e were acting alone. By short-circuiting a condenser (making $E' = 0$) after undergoing absorption and then measuring the residual charge, $c_2 e = D' =$ intrinsic displacement, can be determined, but neither quantity can be determined separately except on the assumption that c_2 is the same for intrinsic as for elastic displacement.

If the spherical electret were placed in a uniform field of intensity E, and if its intensity of electrisation J, or internal displacement D_0, were to remain rigidly fixed (*cf.* § 8), it would be acted upon by the same forcive as that which would act upon the doublet of moment M placed at its center in the same field. This forcive is easily seen to be a torque

$$T = -ME \sin\theta = -\tfrac{4}{3}\pi a^3 JE \sin\theta = -2\pi a^3 D_0 E \sin\theta \tag{16}$$

in the direction of the increase of θ, where θ denotes the angle between the direction of electrisation, or the axis of the doublet, and the direction of the uniform field.

The same result could of course be obtained, though less simply, by integrating the expression $dT = E\sigma' dSx \sin\theta$ over the surface of the sphere, where x denotes the distance from the equatorial plane to the element of area dS of the sphere.

If the sphere is left to itself after the removal of the charges producing the uniform field, the intrinsic displacement, and therefore the internal and external fields, gradually disappear. This

gradual disappearance is due to the gradual diminution of the intrinsic intensity and the continuous action of the self-deëlectrising intensity, which acts against the displacement, or tends to reduce the apparent surface density. Or, better, as the intrinsic forces diminish, the tubes of displacement are gradually *freed* and, being closed tubes, contract to nothing.

If the sphere is covered with a conducting coat while the internal displacement has the value D_0, the external field disappears entirely and the sphere itself is left in the same condition as the dielectric of a condenser short-circuited after absorption has taken place.* There is no potential difference anywhere, but the intrinsic displacement within the sphere has increased, since E', which before opposed the intrinsic force e producing or maintaining the displacement, is now zero. The hemisphere which before had a positive fictitious charge has now a true negative charge, and the other hemisphere, before apparently negative, has now a true positive charge. The law of the distribution of the true charge over the sphere is the same as the law of the distribution of the previous fictitious charge. If the displacement is now denoted by $D' = c_2 e$, (15) gives

$$\begin{aligned} D' = c_2 e = D_0 - c_2 E' &= [(2c_1 + c_2)/2c_1]\, D_0 \\ &= [(2c_1 + c_2)/3c_1]\, J \end{aligned} \qquad (17)$$

where D_0, J, E' are the values of the displacement, intensity of electrisation, and self-deëlectrising intensity immediately before the short-circuiting, and c_2 and e are assumed to remain constant during the process. Since the final value of the self-deëlectrising intensity is zero, the final value of the intensity of electrisation is $J' = D'$. D' and J' are wholly intrinsic, D_0 and J only partially so. The density of the true charge is

$$\sigma = -D' \cos\theta \qquad (18)$$

No change is produced by removing the conducting cover. After its removal, as the intrinsic electrisation continues to di-

* See Heaviside, *Electrical Papers*, Vol. I., p. 491.

minish, an internal and external field of the same character as that of the original field, but opposite in direction, is developed. As the true surface density everywhere remains constant (insulation being supposed perfect), while the intrinsic electrisation gradually diminishes, this field will grow in strength; and if the intrinsic displacement could disappear entirely, which is impossible as long as any electric field remains, the sphere being absorbent, the total field would finally become that connected with the true charges given by

$$\sigma = -D' \cos\theta$$

only, the fictitious charges having entirely disappeared. The displacement and intensity would now be in the same direction at any point within the sphere, as well as without.

6. Infinite Circular Cylindrical Electret Uniformly Electrised Transversely. In exactly the same way, we have in this case for the apparent surface density

$$\sigma' = J\cos\theta = 2\,D_0\cos\theta \tag{19}$$

and for the deëlectrising intensity, or intensity due to the fictitious charges, within the cylinder

$$E' = -J/2c_1 = -D_0/c_1 \tag{20}$$

D_0 and J denoting the internal displacement and intensity of electrisation, respectively.

The external field is that part outside the cylinder of the field of the line doublet of moment

$$M = \pi a^2 J = 2\pi a^2 D_0 \tag{21}$$

placed along its axis.

The plane diagram of the field can be obtained from Fig. 28 by simply drawing a circle of radius a with the center of the diagram as center, annulling all the lines within this circle, and connecting by straight lines the ends of each circular arc.

The magnitude of the fictitious charges on the positive and negative halves of a unit length of the cylinder is

$$q' = 2aJ = 4aD_0 = 2\Pi \tag{22}$$

the flux across unit length of the cylinder is

$$\Pi = 2aD_0 = \tfrac{1}{2}q' \tag{23}$$

and the flux of intensity from or to the fictitious charge upon unit length (either half) is

$$\Pi' = \Pi/c_1 - 2aE' = 2\Pi/c_1 = q'/c_1 \tag{24}$$

7. Natural Electrets. Pyroelectric Crystals. Kelvin's Theory.* A state of intrinsic electrisation exists naturally in certain crystals, for example, tourmaline, which are called, from the thermal relations described below, *pyroelectric* crystals. In its ordinary condition, however, after remaining some time at a constant temperature, the external field of such an electret has disappeared, on account of poor insulation, like that of the spherical electret of § 5 after being covered with a conducting coat. The electret has now a positive charge at one end and a negative charge at the other, terminating the tubes of intrinsic displacement (there is no elastic displacement). Altering the temperature of the electret alters its intrinsic intensity and state of electrisation, and therefore, if the surface remains sufficiently well insulated to retain its charges when some of the tubes of intrinsic displacement become free, develops an external field. The direction of this field depends on the direction in which the intrinsic forces and electrisation alter with the increase or decrease of temperature. In tourmaline, as would be expected in every case, the electrisation decreases with temperature increase. Hence by heating tourmaline a field directed like that of the sphere of § 5, after being short-circuited and then left insulated for a time, is developed. If the insulation is not perfect, this external field will gradually disappear. If the electret is now cooled, its electrisation will increase and an external field opposite to the former field will appear, and then gradually disappear by conduction when the tem-

* See Heaviside, *Electrical Papers*, Vol. I., p. 493.

perature is kept constant. The intrinsic displacement of a pyroelectric crystal, and therefore the true surface charges, cannot be made to disappear like those of an electret whose electrisation is due to absorption. By breaking a pyroelectric substance across its axis of electrisation, however, positive and negative apparent charges, and an external field connecting them, without true charges, may be developed.

Piezoelectric Crystals. A state of electrisation accompanied by internal and external fields similar to those of the sphere of § 3 is produced in some crystals, called *piezoelectric* crystals, by compressing or stretching them, and disappears when the compression or stretch is removed. The external field corresponding to a given state of strain may disappear by surface conduction, leaving the surface with true charges, like the electrets described above. If the state of strain is altered after this condition has been reached, an external field is developed whose direction depends on the direction of alteration of the strain. Some crystals, like tourmaline, are both pyroelectric and piezoelectric.

8. Permanent Electret. The fictitious charge, or pole strength, of half of a symmetrical isolated electret is not, as we have seen in two particular cases, §§ 5 and 6, equal to the flux through the electret, but is greater than this flux. Thus, although the fictitious charge upon half of an originally neutral sphere placed in a uniform field is less than the flux through the sphere, the fictitious charge upon half of an isolated spherical electret uniformly electrised is one and one half times as great as the flux through the sphere ; and the fictitious charge upon half the surface of an isolated infinite cylindrical electret uniformly electrised transversely is twice as great as the flux through the cylinder. The infinite cylinder may be regarded as an ellipsoid of revolution about an infinite axis perpendicular to the direction of the electrisation, and the sphere may be regarded as an ellipsoid with its three axes equal. We shall see below that in the case of a circular cylinder whose length is very great in comparison with its

diameter and which is electrised longitudinally, the pole strength at either end is approximately equal to the flux through the cylinder. As the ratio of the length of the cylinder to its diameter approaches infinity, the ratio of the pole strength of either end to the electric flux through the cylinder approaches unity indefinitely. In the limit we have an ellipsoid of revolution about an infinite axis parallel to the internal displacement. Thus the greater the ratio of the axis parallel to the electrisation to the other axes, the more nearly does the pole strength equal the electric flux across a pole or through the electret.

It is clear that if an electret is brought into the field of an electric charge or another electret, the distribution as well as the strength of each of the electret's poles (or each of the poles of both electrets) will, in general, be altered.

Moreover, if the medium surrounding an electret is replaced, in whole or in part, by a medium of different permittivity, the flux through the electret, and therefore the fictitious charges or pole-strengths, will increase or decrease, as well as change in distribution, according as the permittivity of the new medium is greater or less than that of the old medium. For the displacement, internal and external, is maintained by the intrinsic forces within the electret, which remain constant or appreciably constant during the change, independently of the surrounding medium, and must produce a greater or less flux the greater or less the permittivity. An increase of the same kind, and greater in extent, occurs when the external field is destroyed, the surrounding medium being made conducting (or the permittivity infinite).

The energy of the electret's field may be divided into two parts, the energy within the electret, mostly energy of intrinsic displacement, and the energy of the external medium. It is clear that the greater the ratio of the intrinsic energy of the electret to the energy of its external field, the less will the electrisation of the electret be affected, either in distribution or in amount, by changes in this external field (resulting from changes

in the medium); also the greater the ratio of the energy density of the intrinsic electrisation to the energy density of the external field, the less will the electret be affected by the introduction of other electrets or charges.

Consider now the ideal case of a cylindrical electret of very small cross-section and great length uniformly electrised in the direction of its length, except within very small regions close to its ends in which the flux diverges or converges. The poles are approximately concentrated at the ends of the electret, and, except within the very small volume occupied by the electret itself, the electric field is similar to the field surrounding two electric charges approximately concentrated at a distance apart equal to the length of the electret. If the length of the electret is very great, the external field around each pole is practically radial.

Since the energy density at any point of the surrounding medium is proportional to the square of the intensity, and since the intensity is inversely proportional to the square of the distance from a pole (provided the distance is small in comparison with the length of the electret), the energy of the external medium is confined almost wholly to small regions surrounding the two poles. If the length of the electret is increased while the intensity of its electrisation is kept constant, the external energy will therefore remain very nearly constant.

The energy of the uniform intrinsic electrisation, however, is proportional to the length of the electret for a given value of the internal electrisation or displacement.

Hence by increasing the length of the electret, and keeping the flux and therefore the pole strengths constant, the ratio of the energy of the intrinsic electrisation to that of the external medium may be greatly increased.

Hence such an (ideal) long slender longitudinally electrised electret, if made of a substance with intrinsic energy density very great for a given intensity of electrisation (which would be called an electrically hard substance), would be approximately a *per-*

manent electret, its internal energy, electric flux, poles, and pole strengths, being practically independent of the external field (unless the external field should be destroyed, when, although the internal energy and flux would remain sensibly constant, the pole strengths would be reduced to zero).

The force between such a pole (concentrated) and an extremely small body with (concentrated) charge q distant L therefrom would be

$$F = q\,q'/4\,\pi c L^2 \tag{25}$$

where q' is the pole strength and is constant (at a given temperature), and c is the permittivity of the surrounding medium.

Since the volume of the electret is negligible, and the flux from each pole in the external medium radial, the reaction between the two fields and therefore the force between the pole and the charged body must be the same as the force between a very small body with concentrated *true* charge equal to Π, the flux through the electret, and the small body with concentrated charge q. That is,

$$F = q\Pi/4\pi c L^2 \tag{26}$$

On comparing (26) with (25), we see that

$$q' = \Pi \tag{27}$$

or, the flux through an (ideal) extremely slender longitudinally electrised electret of great length is equal (strictly, sensibly equal) to the fictitious charge, or pole strength, at either end.

It is clear from what precedes that any of the electric fields described in preceding chapters would remain sensibly unaltered if each concentrated true charge were replaced by the concentrated pole of an (ideal) permanent electret of very great length and negligible cross-section, and with pole strength or longitudinal flux equal to the charge replaced.

CHAPTER VII.

SPECIFIC INDUCTIVE CAPACITY. THE COMPARISON OF PERMITTIVITIES.

1. Specific Inductive Capacity. The *specific inductive capacity* of a substance is defined as the ratio of its permittivity to the permittivity of the standard medium. If, as in this book, free æther is chosen as the standard medium, the specific inductive capacity of a dielectric is numerically equal to its permittivity (measured in the electrostatic systems of units, XIV.), since $c_0 = 1$ (in the electrostatic systems).

The Comparison of Permittivities, or the Determination of Specific Inductive Capacity. Four general methods of comparing permittivities will be considered here:

I. The permittance of a dielectric bounded by a fixed system of conductors is proportional to its permittivity. Hence if the whole field is filled in succession with two dielectrics, and the two capacities compared experimentally, the ratio of the permittivities will be known. If the condenser contains two different dielectrics at the same time in one of the experiments, the method may still be used in certain simple cases, with little modification. See §§ 7 and 8, IV.

II. The forcive between two given conductors is proportional to the permittivity of the dielectric filling the field if the voltage is kept constant, and inversely proportional to the permittivity if the charges are kept constant. Hence by keeping the voltage constant and comparing the forcives when the field is filled with two dielectrics in succession the ratio of the permittivities may be determined. The comparison by means of constant charges is in general impracticable. When the field contains two dielec-

trics at the same time, as in § 7, IV., the method is still applicable, with slight modification, in certain simple cases. See § 2.

III. The forcive upon a dielectric of permittivity c_2 bounding, or surrounded by, another dielectric of permittivity c_1, in an electric field depends upon the ratio of c_2 to c_1. Thus by measuring F in § 11, IV., c_2/c_1 may be determined. Two methods based upon this principle, one for liquids, and the other for solids (or fluids contained in a vessel made of a solid dielectric of known permittivity), are described in §§ 3 and 4.

IV. When lines of displacement are refracted in passing from one dielectric to another, $\tan\theta_1/\tan\theta_2 = c_1/c_2$. Hence by measuring θ_1 and θ_2, the ratio c_1/c_2 may be determined. This method is discussed in § 5.

From §§ 1–2, VI., it is clear that the permittivity (if defined as D/E) of most dielectrics depends to a greater or less extent upon the time of electrisation, being greater the greater the time, up to a certain limit, and on the previous history of the dielectric (*cf.* the curve in Fig. 66), except for slow processes.

From §§ 2–5, V., it follows that the permittivity depends to some extent upon the stresses in the dielectric, which may be produced wholly by electrical causes, unless the coefficients there defined vanish.

2. Method II. With the Quadrant Electrometer. Since S, § 5, III., is proportional to c, θ is also proportional to c for given values of V_A, V_B, and V_{AB}. Hence by submerging the quadrants in two dielectrics of permittivities c_1 and c_2 successively and measuring the resulting deflections for the same voltages, we have

$$c_2/c_1 = \theta_2/\theta_1 \tag{1}$$

With the Kelvin Absolute Electrometer or the Bichat and Blondlot Electrometer In the same way, by submerging the conductors of either of these electrometers in two dielectrics in succession and measuring the corresponding values of F, § 4, III., we have, for constant voltages,

$$c_2/c_1 = F_2/F_1 \tag{2}$$

In the case of the Kelvin electrometer the force may be kept the same in the two experiments and the comparison made by altering d. In this case, if d_1 and d_2 denote the values of d when the first and second dielectrics are in the field

$$c_2 / c_1 = d_2^2/d_1^2 \tag{3}$$

By (25), IV. and (3), III. the Kelvin electrometer may also be used when one of the dielectrics is in the form of a plane slab of a given thickness $d_2(< d)$. The equations corresponding to (4), III. and (5), III. are easily developed for this case.

When any one of these instruments is used idiostatically, alternating as well as unidirectional voltages may be applied, and the permittivity thus determined when the time of electrisation is very short.

3. Method III. Quincke's Method for Liquids. Fig. 68 is a diagram of the apparatus. A and B are the conductors of a parallel plate condenser separated by a distance d very small in comparison with their breadth and length, and immersed in a liquid D whose permittivity, c_2, is to be compared with that of a

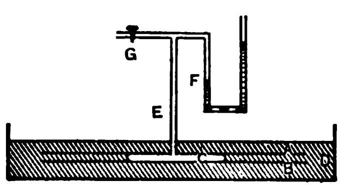


Fig. 68.

gaseous dielectric, as dry air, of permittivity c_1. A tube E communicates through a small opening in A near its center with the region between the two plates. This tube is continuous with a manometer tube F and communicates with a bulb containing dry air by the stop-cock G. The manometer tube contains a liquid of density ρ. Its cross-section will be denoted by A.

In performing an experiment a wide flat air bubble C is first formed between the plates by opening G and pressing the air bulb. G is then closed, and the difference of level between the

two liquid surfaces in F is read. If this difference of level is denoted by h', and if the acceleration of gravity is denoted by g, the excess of the pressure in C (due to the hydrostatic pressure of the liquid D and the capillary pressure inward at the edges of the air bubble) over the atmospheric pressure is $h'A\rho g$. The condenser is now charged to a voltage V. The electric intensity between the plates is uniform, except near their edges and near the edges of the bubble, and equal to $E = V/d$. The electric pressure (§ 41, I.) within the uniform part of the field of the dielectric D is $\frac{1}{2}c_2E^2$, while the electric pressure within the uniform part of the field of the air bubble C is $\frac{1}{2}c_1E^2$. Hence, if c_2 is greater than c_1, the bubble will contract until sufficient air has been forced out into the manometer tube to increase the difference of level by h, and the gaseous pressure by $hA\rho g$, where

$$hA\rho g = \tfrac{1}{2}c_2E^2 - \tfrac{1}{2}c_1E^2$$

when there will again be equilibrium. Hence by observing h, A, ρ, g, and $E = V/d$, $c_2 - c_1$ may be obtained from the equation

$$c_2 - c_1 = 2hA\rho g/E^2 = 2hA\rho gd^2/V^2 \qquad (4)$$

In what precedes we have assumed the capillary pressure of the bubble and the hydrostatic pressure of the liquid D, as well as the total volume of the air, to remain constant throughout the experiment. We have also assumed the slight alterations of gaseous and liquid pressures occurring during the experiment to bring about no alterations of the permittivities. The fact that these conditions are not exactly fulfilled will evidently introduce no sensible error.

(4) Can be deduced also by the method of § 55, I., from energy considerations.

4. Method III. for Solid Dielectrics. We shall consider only the simplest case, when the dielectric is in the form of a plane slab. Fig. 69 is a diagram of the apparatus. A and B are the two conductors of a parallel plate condenser separated by a dis-

tance d small in comparison with their length and breadth. C is a plane slab of the dielectric, of permittivity c_2, hanging in air, of permittivity c_1, with its sides parallel to A and B, from the arm of a balance D. The thickness of the slab will be denoted by d_2 and its width (perpendicular to the plane of the paper) by L.

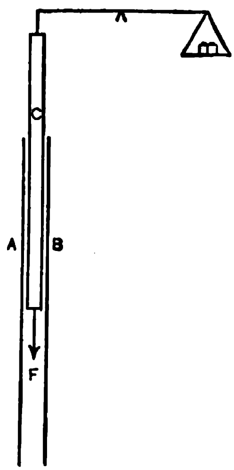


Fig. 69.

C is first balanced by adding weights to the scale pan of D while A and B are at the same potential. A and B are then charged to a voltage V. This will produce a straight field between A and B, except near their edges and near the edges of the slab C, and will disturb the equilibrium. Equilibrium is then restored by adding weights to the scale pan if c_2 is greater than c_1, or by removing weights therefrom if c_2 is less than c_1, as follows from §§ 7 and 10, IV., or from § 7, IV., and § 55, I. Let F denote the downward force upon C due to the charging of the condenser. F can be found at once by the method used in § 3, or can be determined as follows by the method of § 55, I.

Imagine C to suffer an infinitesimal displacement dx downward from its equilibrium position. This will increase the cross-section of the uniform part of the field through C and air by Ldx, and will diminish the cross-section of the uniform part of the field passing through air only by the same quantity. The energy of the weak field outside the condenser and that of the non-uniform field near the edges will remain sensibly constant, and the non-uniform field at the edges of C will move unaltered with C. Hence the only appreciable change in the energy of the field is that due to the fact that the cross-section of the uniform field with two dielectrics has increased by Ldx, while that of the uniform field through air only has decreased by the same quantity. Hence the total increase in energy is sensibly

$$dW = \tfrac{1}{2}c_1 V^2 Ldx/\{d - [(c_2 - c_1)/c_2]d_2\} - \tfrac{1}{2}c_1 V^2 Ldx/d$$
$$= \tfrac{1}{2}c_1 V^2 (c_2 - c_1)d_2 Ldx/\{d[c_2 d - (c_2 - c_1)d_2]\} = Fdx$$

Hence

$$F = dW/dx = Lc_1(c_2 - c_1)V^2/\{2d[c_2 d - (c_2 - c_1)d_2]\} \quad (5)$$

When L, c_1, F, V, d, and d_2 are known, c_2 can be determined from this equation.

In obtaining this result we have assumed the field uniform throughout the whole length L, except near the lower and upper edges of the slab. To make the error arising from the non-fulfilment of condition negligible L must be great in comparison with d.

5. The Method of Refraction of Lines of Displacement. (Perot, *Comptes Rendus*, 113, p. 415, 1891.) To compare two permit-

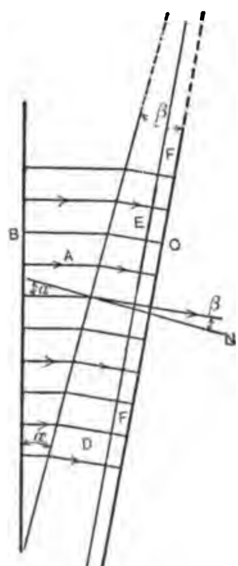


Fig. 70.

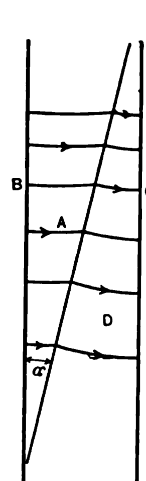


Fig. 71.

tivities by this method it is necessary, as in §§ 3 and 4, that one of the dielectrics be a fluid, as air. The principle of the method may be developed as follows.

Let a large triangular prism A, Fig. 70, of permittivity c_2 and with angle α be placed with one of its plane sides in contact, or otherwise parallel, with a large metallic plane B; and let another large metallic plane C, separated from A (or A and the other conductor) by a fluid dielectric D, as air, of permittivity c_1, be arranged so that the angle β between C and the nearer face of A can be varied and measured. If B and C are charged to a voltage V, the electric field will not, in general, be uniform either in A or in D. The approximate field when B and C are parallel is shown in Fig. 71. But evidently there will always be a certain value of β for which the field in A and also the field in D will be uniform, except near the edges, as shown in Fig. 70. In this case $\theta_1 = \alpha$ and $\theta_2 = \beta$ (§ 2, IV.). Hence

$$c_2/c_1 = \tan\beta/\tan\alpha \qquad . \qquad (6)$$

To find this position, a very thin metal plate E is so attached to fine insulating threads FF as to be movable *with its plane parallel to C only*. If the field in D is uniform, this motion will not disturb the voltage between the plates; otherwise the voltage will be altered. To make the test, then, C is connected to earth (the walls of the room) while B is charged to potential V (potential of earth = 0). Then the condenser is insulated and C is connected with the electrode of an electrometer, the other pole of which is to earth. Then E is displaced while kept parallel to C. If the electrometer still indicates that the potential of C is zero, the field in D, as well as that in A, is uniform. If not, a second adjustment of β must be made, and tests and adjustments repeated until the potential of C remains zero, or until the disturbance of its potential is a minimum, when E is displaced. Then c_2/c_1 can be determined from (6).

The close agreement between the values of c_2/c_1 found by this method and the same ratio determined by other methods serves to verify the correctness of (3), IV., on which the method is based.

CHAPTER VIII.

THE ELECTRIC CURRENT. THE CONDUCTION CURRENT.

1. The Convection Current. If a small light conductor, such as a gilded pith ball, is suspended by a long insulating thread between the vertical plates A and B of a charged parallel plate condenser, it will fly back and forth between the plates carrying opposite charges in opposite directions and gradually discharging the condenser.

The rate dq_1/dt at which positive charge is carried from A to B, or the rate dq_2/dt at which negative charge is carried from B to A, or the sum of the two rates, $dq_1/dt + dq_2/dt$, if both processes occur simultaneously (as would be the case if several pith balls were present), is called the *electric convection current* from A to B, and will be denoted by I_{cv}. That is,

$$I_{cv} = dq_1/dt + dq_2/dt = dq/dt \tag{1}$$

Strictly, the convection current is limited to the space actually occupied by the moving charge or charges.

If the electric volume density of the positive charge in the element of volume at a given point is ρ_1, and the density of the negative charge in the element ρ_2 (the element containing, in general, both positive and negative charges), and if the velocity of the positive charge is u_1 and that of the negative charge in the opposite direction u_2, then the convection current per unit area across a surface normal to u_1 in the element of volume, or the *electric convection current density* in the element, is

$$i_{cv} = \rho_1 u_1 + \rho_2 u_2 \tag{2}$$

in the direction of u_1.

2. The Dielectric Current. If the electric flux Π across a surface is increasing at the rate $d\Pi/dt$, there is said to be a *dielectric current* or an *electric displacement current* through the surface equal to

$$I_d = d\Pi/dt \tag{3}$$

If the displacement D at any point is changing at the rate dD/dt, the *dielectric current density* at the point is

$$i_d = dD/dt \tag{4}$$

which is evidently a vector with the direction of dD.

According to our mechanical conception, § 14, I., the dielectric current in free æther or material insulators would be a convection current of æther cells or corpuscles.

Convection and dielectric currents will be more fully discussed in a later chapter (XV.). The remainder of this chapter will be devoted principally to the conduction current.

3. The Conduction Current. If two condenser plates A and B with positive and negative charges, respectively, are connected by a wire M, the electric field will disappear by the process described in § 42, I. During this process there is an electromotive force *along and through* the wire in the direction AMB, and the wire is traversed in part by positive charges in the direction AMB and in part by negative charges in the opposite direction BMA. The wire is said to be traversed by a *conduction current.* As we shall see later (IX., § 15) there is reason to believe that the conduction current consists, in the general case, in a stream of positively electrified *particles* in the direction of the e.m.f. and a stream of negatively electrified *particles* in the opposite direction across every section of the conductor.

We shall define the *strength* of the conduction current, or the *conduction current*, I_c, across any section of the conductor as the rate at which electric charge is transferred across that section; and we shall define the direction of the current across the section as the direction in which the positive charge is carried across

the section, or the direction opposite to that in which the negative charge is carried. If both positive and negative charges are simultaneously crossing a given section in opposite directions, and if dq_1 and dq_2 are the magnitudes of the positive and negative charges carried across in the time dt, the conduction current across the section is

$$I_c = dq/dt = dq_1/dt + dq_2/dt \qquad (5)$$

in the direction of transfer of the positive charge.

When the charges or the induction in (1), (3), or (5) are expressed in *RES* units and the time in seconds, the electric current is said to be expressed in the *RES unit current.*

A method of measuring the electric conduction current based on the above definitions is described in § 3, IX.

In the case of an ordinary condenser system the phenomenon of discharge, or the electric current, lasts only a small fraction of a second. In a variety of ways this time may be increased; and if the ends of the wire M, instead of being connected to the plates of a condenser, are joined to the terminals of a voltaic cell, or other agent capable of maintaining the voltage between its ends constant, transient effects similar to those described in § 42, I., will at first occur, but a *steady* or unchanging state will soon set in.

4. The Conduction Current Density. If a small plane area dS is imagined within the substance of a conductor carrying a current, it is obvious that the quantity of electric charge crossing dS per second will be different when dS is turned in different directions, and will be a maximum when the normal to dS points in the actual direction of transfer of charge at the point. The ratio of the current dI_c crossing the area dS, with its normal turned in this direction, to the area dS, is a vector called the *electric conduction current density* at the point considered, and will be denoted by i_c. Thus

$$i_c = dI_c/dS \qquad (6)$$

If for the subscript c we substitute cv or d, (6) will define the

convection or displacement current density in terms of the convection or displacement current consistently with §§ 1 and 2.

5. Kirchhoff's Law I. When an electric conduction current is steady, it flows in a closed circuit and has the same value at every section of the conductor which carries it. For if the circuit were not closed, or if the current across any two sections were not the same, a positive or negative electric charge would continually accumulate at either end or in the region between the two sections, and the state would therefore not be steady.

This proposition will be extended to the general electric current in § 8, Chapter XV.

An obvious (but very incomplete) mechanical analogue of the steady conduction current is the flow of an incompressible liquid through an endless pipe.

Kirchhoff's law I., applied to the unit volume of a conductor carrying a steady current, may be written

$$\operatorname{div} i_c = \operatorname{conv} i_c = 0 \tag{7}$$

since the current entering any element of volume across one part of its surface is equal to the current leaving the element across the rest of the surface.

Stream-tubes and Stream-lines. From (6) and (7) it is now evident that a steady current within a conductor may be mapped out by a system of lines and tubes analogous to lines and tubes of intensity, etc. These tubes and lines are called *stream-tubes* and *stream-lines*, or *lines* and *tubes* of *current* or *flow*. The current density at any point has the direction of the line of flow through that point; and the strength of the current across every section of a given tube is the same and equal to $\int i_c dS$ over a diaphragm S normal to the stream lines.

In what follows we shall drop the subscript c and denote the conduction current and current density by I and i.

6. Electrodes. Two equipotential surfaces across one of which the current enters a conductor and across the other of which the

current leaves the conductor are called the *electrodes* of the conductor. The electrode by which the current enters the conductor is called the *anode*, and that by which the current leaves is called the *kathode*.

7. Ohm's Law for a Steady Current in a Homogeneous Isotropic Conductor at Uniform Temperature. Along a homogeneous isotropic conductor at uniform temperature throughout, let a steady current I flow, and let the corresponding voltage between two electrodes in or terminating the conductor be denoted by V_{12}. Then, as a result of experiment, it may be stated that, if the voltage or current is varied while the temperature is kept constant (and in some substances at least certain other physical conditions), the current I is proportional to the voltage V_{12}. That is,

$$I = KV_{12} = V_{12} / R \tag{8}$$

The proportionality factor K is called the *conductance* of the portion of the conductor between the given electrodes, and its reciprocal, $R = 1 / K$, the *resistance* of the conductor. The quantities on which R and K depend will be discussed below.

Definitions of the *RES* unit conductance and resistance follow in the usual manner from the above equations. Thus when I and V_{12} are expressed in *RES* units, the conductance and resistance also are said to be expressed in *RES* units.

(8) expresses the integral form of *Ohm's law* for homogeneous isotropic conductors. The expression of the law for non-homogeneous circuits, etc., will be developed in following articles.

8. Conductance and Resistance of a System of Conductors Connected in Multiple. If the electrodes of any number, n, of conductors are joined together so that the anodes form a common anode and the kathodes a common kathode, the conductors are said to be connected in *multiple*. Let $K_1, K_2, \cdots, K_n$ and $R_1, R_2, \cdots, R_n$ denote the individual conductances and resistances of the conductors, $I_1, I_2, \cdots, I_n$ the individual currents, and I the total current through all the conductors, when the voltage be-

tween the electrodes is $V_1 - V_2$. Then the conductance of the system is

$$K = I/V_{12} = (I_1 + I_2 + \cdots + I_n)/V_{12}$$
$$= K_1 + K_2 + \cdots + K_n \tag{9}$$

and the resistance of the system is

$$R = 1/K = 1/(K_1 + K_2 + \cdots + K_n)$$
$$= 1/(1/R_1 + 1/R_2 \cdots + 1/R_n) \tag{10}$$

9. Conductance and Resistance of a System of Conductors Connected in Series. If any number, n, of conductors is connected up end to end in such a manner that each surface of contact between two conductors is an equipotential for both and coincident with the original electrodes of the conductors when separate, the conductors are said to be connected in *series*. Let V_{0n} and V_{01}, V_{12}, $\cdots$, $V_{(n-1)n}$ denote the voltage between the terminal electrodes and the voltages between the ends of the successive conductors, and let I denote the current along all the conductors. Then the conductance of the system is,

$$K = I/V_{0n} = 1/(V_{01}/I + V_{12}/I + \cdots + V_{(n-1)n}/I)$$
$$= 1/(1/K_1 + 1/K_2 + \cdots + 1/K_n) \tag{11}$$

and its resistance is

$$R = 1/K = R_1 + R_2 + \cdots + R_n \tag{12}$$

10. The Cylindrical Homogeneous Isotropic Conductor. Resistivity and Conductivity. Suppose n precisely similar right cylindrical conductors joined in multiple with their ends as common anode and kathode, thus making a cylindrical conductor of n times the cross-section of each of the original conductors. If R_0 and K_0 denote the resistance and conductance of each cylinder separately, and R and K the resistance and conductance of the system of n conductors in multiple, that is, of a cylinder of

n times the cross-section with its ends as electrodes, it follows from (9) and (10) that

$$K = nK_0 \quad \text{and} \quad R = R_0/n \tag{13}$$

Thus the resistance of a cylindrical conductor of constant length with its ends as electrodes is inversely proportional to its cross-section, or its conductance is directly proportional to its cross-section.

Suppose the n conductors connected up in series, thus making a cylindrical conductor of the original cross-section, but of n times the original length. If R denotes the resistance of the system and K its conductance, (11) and (12) give

$$K = K_0/n \quad \text{and} \quad R = nR_0 \tag{14}$$

Thus the resistance of a cylindrical conductor with its ends as electrodes is proportional to its length, or its conductance is inversely proportional to its length.

Putting the two above results together, we see that the resistance R of a cylindrical conductor with its ends as electrodes is proportional to its length and inversely proportional to its cross-section, or that its conductance K is inversely proportional to its length and directly proportional to its cross-section. That is, if L denotes the length of the cylinder and A its cross-section,

$$K = kA/L \quad \text{or} \quad R = rL/A \tag{15a}$$

where r and k are constants depending on the chemical constitution of the conductor and its physical condition. r is called the *specific resistance* or *resistivity* of the substance, and k, its reciprocal, is called the *conductivity* of the substance. (15) may be written

$$r = RA/L \quad \text{or} \quad k = KL/A \tag{15b}$$

When R and K are expressed in RES units and A and L in c.g.s. units, r and k are said to be expressed in the *RES units resistivity* and *conductivity*. r is equal, in magnitude, to the re-

sistance of a cube of the substance with unit edge when two opposite faces are electrodes.

11. Differential Form of Ohm's Law. From the preceding article and the principle of symmetry it follows that in the case there considered the electric equipotential surfaces *in the conductor* are parallel to the electrodes, and the electric intensity E uniform and parallel to the length of the conductor.

From the same article it also follows that the current density is uniform throughout the conductor, and the lines of flow parallel to the length of the cylinder, and therefore coincident with the lines of intensity.

Hence we may substitute in (8) for V_{12} its equal EL, for I its equal Ai, and for K its equal kA/L; then $Ai = kA/L \cdot EL$, or

$$i = kE = E/r \tag{16}$$

In the case considered therefore the current density has the same direction as the electric intensity, and is proportional to it in magnitude. Since, moreover, any conductor carrying a steady current may be divided up into elementary tubes of intensity and by equipotential surfaces infinitesimally distant apart, and since the cylindrical volume within one of these tubes between two successive equipotentials is in exactly the same state as any tube of the cylinder considered above, (16) is seen to hold in general whether the tubes are straight or not.

Since at the surface of a conductor the current density is tangential to the surface, the electric intensity within the conductor at the surface is also tangential.

12. The Electric Field of the Steady Conduction Current. Except as stated below, the electric field in the dielectric surrounding a conducting system traversed by a steady electric current has all the properties of a purely static field connected with static charges only. The tubes of displacement terminate at the surfaces of the conductors (if homogeneous) and the surface charges at their ends are not in motion and take no part in the conduction.

For within the conductor (supposed homogeneous) div $i =$ div $kE = k$ div $E = 0$. Hence div $E = 0$; and, since the conductor is homogeneous, div $D = \rho$ must also be zero, whatever the relation of D to E may be. Hence within the conductor the positive and negative charges per unit volume at any point are equal. Thus no tubes from the dielectric penetrate into the conductor, but all end at its surface. That the external field is static and that the surface charges take no part in the conduction, or do not move, follow from the consideration that the field surrounding a conducting system traversed by a steady current can be altered in any manner, by moving the circuit or by bringing up charged bodies insulated from it, without in any way affecting the (steady value of the) current. Also the surfaces of insulated conductors placed in the field are equipotential surfaces and traversed by no currents; hence the tubes ending upon them (and connected with the current-carrying conductors) are not in motion. The same thing follows from Ohm's law, the resistance of a conductor not being a function of its external surface, as it would be if the surface charges took part in conduction.

The conductor itself, as shown in § 11, also contains an electric field invariable with the time. Little or nothing is known of the electric displacement in good conductors traversed by steady currents.

The electric field within and without the conductor is accompanied by a magnetic field (XI. and XII.) and is the seat of the transfer of energy (XVI.), while the conductor is also the seat of the dissipation of energy in heat (§ 15).

The lines of intensity within the conductor are tangential at the surface, as shown in § 11, and the lines of intensity in the dielectric do not meet the conducting surfaces normally.

Let E_1 denote the electric intensity in the dielectric just outside the conductor at a point P of the interface, and θ_1 the angle made by E_1 with the normal to the surface of the conductor; and let E denote the intensity just within the conductor at the same point of the interface. E, as already shown, is parallel to the

interface. By the principle of the conservation of energy we may show, just as in § 2, IV., that E_1 and E are in the same plane normal to the interface, and that

$$E_1 \sin \theta_1 = E \tag{17}$$

If the conductor is a *perfect* (imaginary) conductor, that is, if its conductivity is infinite, $E = 0$, since otherwise the current would be infinite. Hence in this case $\theta_1 = 0$, or the tubes of displacement in the dielectric meet the surface of the conductor normally.

13. The Laws of Refraction of Stream-lines. At the interface between two substances of different conductivities k_1 and k_2 a stream-line of a steady current is refracted in such a manner that the incident and refracted lines are in the same plane perpendicular to the interface, and that

$$\tan \theta_1 / \tan \theta_2 = k_1 / k_2 \tag{18}$$

where θ_1 and θ_2 denote the angles made with the normal to the interface by the incident and refracted portions of a stream line.

For, since the current is steady, so that no electric charge accumulates anywhere, Kirchhoff's law I. gives

$$i_1 \cos \theta_1 = i_2 \cos \theta_2 \tag{19}$$

and the principle of conservation of energy gives

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \tag{20}$$

and these equations, since $i = k E$, give (18) on division. Cf. § 2, IV.

c being interchanged for k and i for D, the discussion in § 2, IV., and the description in following articles of fields in two or more dielectrics apply to the fields of intensity and flow in conductors. It must always be remembered, however, that one of the k's may be zero, while c can never be less than $c_0 = 1$; so that k_2/k_1 may be zero or infinity without either k_1 or k_2's being infinite.

14. General Formula for Conductance and Resistance. Conductance and Permittance. From (8) we have

$$K = 1 / R = I / V_{12}$$

while (6) gives

$$I = \int i dS$$

the integration being taken over an electrode, or over the portion within the conductor of any equipotential surface. Moreover,

$$V_{12} = \int E dL$$

along a line of intensity or flow from one electrode to the other. Hence

$$K = 1 / R = I / V_{12} = \int kEdS / \int EdL = \int kEdS / V_{12} \quad (21)$$

By comparing (16) and (21) with (3) and (24), Chapter I., it will be seen that K bears the same relation to k that the permittance S bears to the permittivity c. Since $K = \int kEdS / V_{12}$, while $S = \int cEdS / V_{12}$, the process of finding the conductance of the portion of a conductor between two given electrodes or equipotential surfaces is identical with that of finding the permittance of a dielectric occupying the same space as that occupied by the conductor and having the same electric field as that within the conductor, except that k must be substituted for c. In most permittance problems it is impossible to deal accurately with finite electrodes and finite electric fields, there being no substance of zero permittivity with which an electric field may be surrounded to prevent its spreading indefinitely. A conductor, on the other hand, may easily be placed in a region of zero conductivity, so that the current tubes are wholly restricted to its own substance, and within the conductor the lines of flow and lines of intensity are coincident. This makes conductance problems much simpler in many cases than the corresponding problems in electrostatics.

15. The Conductance and Resistance of Various Conductors. In all the examples which follow the electrodes may be supposed

to be surfaces of infinitely thin sheets of *perfectly* conducting material ($k =$ infinity), in order to insure their being true electrodes, that is, equipotentials normal to the lines of intensity and flow (see § 12).

The corresponding permittances having been already determined (Chapter II.), the conductances are found as indicated in the last article, from the following relation

$$K = kS/c \qquad (22)$$

1. For a right cylinder of cross-section A and length d, with its ends as electrodes, (29), II., gives, with (22),

$$K = 1/R = kA/d \qquad (23)$$

as already shown directly in § 10.

2. For a conducting spherical shell between two spherical electrodes of radii L_1 and $L_2 = L_1 + d$, (5), II. gives, in the same way,

$$K = 1/R = (4\pi k/d)\, L_1^2\, (1 + d/L_1) \qquad (24)$$

For a hemispherical shell, half of the last, we have

$$K' = \tfrac{1}{2}K = (2\pi k/d)\, L_1^2\, (1 + d/L_1) \qquad (25)$$

and so on for all fractions of the shell obtained by cutting it up with cones having their apices at its center.

3. For the conductance of an infinite conductor in which two spherical electrodes of radii L_1 and L_2 are immersed at a great distance apart, a slight generalisation of (43), II. gives

$$K = 4\pi k/(1/L_1 + 1/L_2) \qquad (26)$$

If the infinite conductor is bounded by a plane surface, and if two *hemispherical* electrodes are placed in the conductor with their bounding circles in the plane of the conductor's surface, the conductance is

$$K' = \tfrac{1}{2}K = 2\pi k/(1/L_1 + 1/L_2) \qquad (27)$$

4. The conductance of a right circular cylindrical shell of radii L and $L + d$ and of length l is, from (23), II.,

$$K = 2\pi kl/\log\,(1 + d/L) \qquad (28)$$

5. The conductance of an infinite plane slab of thickness l and with two right circular cylindrical electrodes of radius R and centers distant $2d$ apart, is, by (62), II.,

$$K = \pi k l / \log \{R / [d - (d^2 - R^2)^{\frac{1}{2}}]\} \tag{29}$$

If the slab is cut symmetrically into halves by a plane passing at right angles to the plane through the axes of the two electrodes, and if this plane is made an electrode, the conductance of either half is twice that of the whole; or

$$K' = 2K = 2\pi k l / \log \{R / [d - (d^2 - R^2)^{\frac{1}{2}}]\} \tag{30}$$

In the same manner, from Chapter IV., the conductances of some simple non-homogeneous conductors may be obtained.

16. Joule's Law: In a homogeneous conductor of resistance R traversed by a current I heat is developed at the rate

$$dH/dt = RI^2 \tag{31}$$

This relation may be established as follows: In the time dt a charge $dq_1(+)$ and a charge $dq_2(-)$ equivalent to a charge Idt $(+)$ in the direction of the current cross every section of a conductor carrying a steady current I. Hence the work done in the time dt by the electric field upon the conductor of resistance R, if the voltage between its electrodes is V_{12}, is

$$dW = V_{12} I dt = RI \cdot I dt = RI^2 dt$$

by §12 and the law of Ohm. Hence the time rate at which work is done by the electric field in the conductor of resistance R is

$$dW/dt = V_{12} I = RI^2 \tag{32}$$

Now heat is always developed in a conductor during the passage of a current. Hence, if no other transformation of energy occurs between the electrodes of the conductor $dW/dt = RI^2$ must be the rate at which heat is generated in the conductor when traversed by the current I. That this is the case

when the conductor is homogeneous (including constancy of temperature throughout) has been proved by the researches of Joule and others, whose experimental results are in strict accord with the above equation when dW/dt is equated to dH/dt. Hence the law, expressed in (31). If the conductor is not homogeneous (and for all but three directions of the stream-lines, if the conductor is not *isotropic*), it contains intrinsic e.m.f.s, § 19, and *in addition* to the Joulean heat transformation other transformations take place.

Differential Form of Joule's Law. The *dissipativity* at any point of a conductor is the time rate per unit current (squared) at which heat is there generated per unit volume. To find the dissipativity, which will be denoted by dh/dt, consider the elementary volume enclosing the given point and included within a tube of flow, of cross-section dS, between two equipotentials distant dL apart. The resistance of this element of volume $d\tau = dL\ dS$, is EdL/idS the current along the tube being idS. Hence

$$dh/dt = (EdL/idS)(idS)^2/(dSdL) = Ei = kE^2 = ri^2 \quad (33)$$

which expresses Joule's law as applied to the element of volume at any point of a conductor.

Experiment justifies the statement that (31) and (33) are applicable to any conductor, homogeneous or not, the total heat developed being equal to the sum of the Joulean heat and the heat developed owing to the operation of other factors than resistance.

By the last equation, (31) may be written

$$dH/dt = RI^2 = \int Eid\tau = \int kE^2 d\tau = \int i^2/k \cdot d\tau = \int ri^2 d\tau \quad (34)$$

the integrals being extended throughout the part of the conductor considered. From this equation it is clear that the amount of heat dissipated per unit time by the resistance of a conductor can be obtained from the formula for the energy in the corresponding case in electrostatics, viz., $W = \int \frac{1}{2}cE^2 d\tau$, by substituting k for $\frac{1}{2}c$, etc.

17. Definition of Resistance by Joule's Law. The proportionality between dH/dt and I^2 having been established by experiment, the resistance of a conductor, R, might have been defined by the relation

$$R = (dH/dt)/I^2 \tag{35}$$

without recourse to Ohm's law. This procedure would be in perfect harmony with all that is known of the nature of resistance, whose only function seems to be the dissipation of energy in heat, as it is dissipated by mechanical friction.

Joule's Method of Determining Resistance in Absolute Measure. By placing a conductor in a calorimeter and measuring the rate dH/dt at which heat is developed therein when a known current I traverses the conductor, its resistance R may be obtained from (35).

18. Mechanical Analogue of the Law of Ohm and the Law of Joule. Consider a pipe through which an incompressible liquid flows at a constant rate, the volume of liquid carried per unit time across every section of the pipe being I. The flow of the liquid is opposed by a frictional pressure assumed to be proportional to I. Let this pressure be denoted by $-RI$, R being a constant for the given pipe and liquid. To overcome this pressure, that is, to keep up the constant rate of flow I, an equal and opposite pressure $V_{12} = +RI$ must be applied in the direction of the current. This pressure does work against friction at the rate $V_{12}I = RI^2$, which is therefore the rate at which energy is dissipated in heat in the circuit.

19. Intrinsic and Impressed Electromotive Force. In order to maintain an electric current, with its continual dissipation of energy in heat according to Joule's law, and its possible performance of work of various kinds, every circuit continuously carrying a current must contain one or more regions in which *energy in some other form*, as mechanical, chemical, or thermal energy, *is transformed into the energy of the electric current* (the energy of the electromagnetic field). Such a region is said to

contain, or to be the seat of, an *intrinsic electromotive force*; and the agent through which the energy transformation is effected, or may be effected, as a dynamo, a voltaic cell, or a thermo-couple, is said to possess the intrinsic electromotive force.

In strictness, the *intrinsic electromotive force* in a region is defined as the rate at which energy in some other form is there *transformed* into the energy of the electric current (the energy of the electromagnetic field) divided by the strength of the current. Thus, if P denotes the rate at which electrical energy is *generated*, or power taken *into* the circuit in the region *by transformation*, I the current, and Ψ the intrinsic electromotive force

$$\Psi = P/I \tag{36}$$

Any part of a circuit in which *electrical energy is transformed into energy of another form* is also said to contain an intrinsic e.m.f., *provided* that the agent effecting the transformation when acting independently can reverse the direction of the transformation, or transform energy of the other form into electrical energy, *i. e.*, itself maintain an electric current. Thus an electric motor, by which electrical energy is transformed into mechanical energy, and a storage battery while charging, by which electrical energy is transformed into chemical energy, possess intrinsic e.m.f.s, since each acting alone can generate electrical energy, the one when mechanically driven acting as a dynamo, the other as a voltaic cell. The intrinsic e.m.f. Ψ is given in all cases by (36), proper attention being paid to the sign of P. Thus if in any region power is taken *into* the circuit by transformation, P in this region is *positive* and Ψ and I have the same direction. If in any region electrical energy is transformed into some other form of energy, or power given *out* by the electrical system, P in this region is *negative*, Ψ and I have opposite signs, or directions, and the intrinsic e.m.f. opposes the current. It is by overcoming this *counter e.m.f.* that the transformation of electrical energy into energy of some other form is effected. In all cases a reversal of the current reverses the sign of the energy transformation by an agent with an intrinsic e.m.f.

Also, *any* region in which energy is transformed from some other form into electrical energy, or from electrical energy into energy of some other form, at the time rate P when the current has the value I is said to contain an *e.m.f.* P/I, although in the latter case the e.m.f. may not be intrinsic. Thus, according to Joule's law, a homogeneous conductor of resistance R when traversed by a current I transforms electrical energy into heat at the rate RI^2. Hence the conductor is the seat of an electromotive force equal to

$$-(dH/dt)/I = -RI^2/I = -RI = -V_{12} \qquad (37)$$

This negative, or counter, e.m.f., $-RI = -V_{12}$ is not, however, included among intrinsic e.m.f.s, since if the conductor is heated a current is not produced, or if the direction of the current is reversed, heat is still *generated* at the same rate, *not absorbed*, as it would be if the e.m.f. were reversible and intrinsic.

Other electromotive forces exist, like potential differences (non-intrinsic e.m.f.s of static fields or fields of conductors carrying steady currents) and the non-intrinsic e.m.f.s of induction (XIII.), by which energy is *transferred* in the electromagnetic field, but never transformed. These e.m.f.s, together with intrinsic e.m.f.s, may act as *impressed* e.m.f.s. The *impressed e.m.f.* between the electrodes of a conductor is equal, by definition, to the sum (§ 21) of the intrinsic e.m.f.s included between them, P/I, *plus* the time rate P', at which electromagnetic energy is *transferred* to the region between them from the surrounding field (developed by *intrinsic* e.m.f.s in other parts of the circuit or in other circuits *transforming* energy of another kind into electromagnetic energy if P'/I is positive) divided by the current I.

Thus the impressed e.m.f. in a homogeneous conductor of resistance R carrying a steady current I is $V_{12} = RI$ (exactly equal and opposite to the counter e.m.f. of resistance, $-RI$), the difference of potential V_{12}, by which the energy is transferred to the conductor at the rate $V_{12}I = RI^2$, being developed by an intrinsic e.m.f. situated outside the portion of the circuit constituting the homogeneous conductor considered. See XVI.

A general characteristic of non-intrinsic e.m.f.s is that they have, like the force of friction in mechanics (analogous to the counter e.m.f. of resistance, $-RI$), or like the elastic reaction of a stretched spring (analogous to a difference of potential), no independent existence of their own, but are developed only through the action of an agent possessing a true intrinsic electromotive force, such as a dynamo or a voltaic cell. This is equivalent to the statement that electrical energy is never *generated* at the expense of energy in some other form through the agency of a non-intrinsic e.m.f.

In what follows many cases of intrinsic and impressed e.m.f.s will be considered.

20. Intrinsic Electric Intensity and E.M.F. The intrinsic e.m.f. in a region may be regarded as the line integral of an *intrinsic electric intensity* in the region. If this intensity is denoted by e, we have therefore

$$\Psi = \int e \cdot \cos \theta \cdot dL \tag{38}$$

where θ denotes the angle between the direction of e and that of the element of the path, dL, at any point.

Consider a tube of flow of cross-section dS at a point where the intrinsic intensity is e. If θ denotes the angle between the directions of e and of i, the intrinsic e.m.f. between two right cross-sections of this tube distant dL apart is $e \cos \theta \cdot dL$. The current through the tube is $i \cdot dS$. Hence the rate at which power is transformed into the circuit per unit volume at the point is

$$dP/d\tau = e \cdot \cos \theta \cdot dL \cdot idS/dSdL = ei \cdot \cos \theta \tag{39}$$

From this power equation $e \cdot \cos \theta$, the component of e in the direction of i, might be defined as

$$e \cdot \cos \theta = (dP/d\tau)/i \tag{40}$$

If P denotes the total power transformed into electrical energy in an isolated electric circuit, and I the current, then, since

$P = dW/dt$ and $I = dq/dt$, $\Psi = P/I = dW/dq$. That is, Ψ is the work per unit charge done in carrying a charge around the circuit. Hence our definition of an intrinsic e.m.f., and therefore our definition of an intrinsic intensity, is in agreement with the general definition of § 17, I. The same is true of the impressed e.m.f.

When P is expressed in ergs per second, and I in the *RES* unit current, or when dW is expressed in ergs and dq in the *RES* unit charge, Ψ is, by definition, expressed in the *RES unit e.m.f.*

21. Intrinsic Electromotive Forces in Series. If any number, n, of agents with individual electromotive forces $\Psi_1, \Psi_2, \cdots, \Psi_n$ are connected up in *series*, so that the same current traverses each, the resultant e.m.f. is

$$\Psi = \Psi_1 + \Psi_2 + \cdots + \Psi_n \tag{39}$$

proper attention being paid to signs. For if P, P_1, P_2, etc., denote the power supplied to the circuit by the resultant e.m.f. and the powers supplied by the individual e.m.f.s, and I the current,

$$P = \Psi I = \Psi_1 I + \Psi_2 I + \cdots + \Psi_n I$$

from which (39) immediately follows.

The above proposition, demonstrated for intrinsic e.m.f.s, is obviously also true for the more general *impressed* e.m.f.s.

Intrinsic Electromotive Forces in Multiple. If any number, n, of similar agents having the same e.m.f. Ψ' are connected up in *multiple*, so that one nth of the current traverses each in the same direction, the resultant e.m.f. Ψ is eqnal to Ψ'. For

$$\begin{aligned} P &= \Psi I = P_1 + P_2 + \cdots + P_n \\ &= \Psi' I/n + \Psi' I/n + \cdots + \Psi' I/n = \Psi' I \end{aligned} \tag{40}$$

22. Ohm's Law, General Form, Deduced from Joule's Law. Let the resistance of a conductor $1C2$, Fig. 72, between two electrodes 1 and 2 be denoted by R. Let the conductor be traversed by a current, reckoned positive when in the direction $1C2$, with the

same value I at any instant across every section, and let the conductor be acted upon by an impressed e.m.f. $\Psi_{12} + \Psi_{12}' = P/I + P'/I$, Ψ_{12} and Ψ_{12}' being reckoned positive when in the direction of the current I (*i. e.*, $1C2$). Ψ_{12} is the intrinsic e.m.f. between the electrodes, and Ψ_{12}' the electromagnetic energy per unit current *transferred* per unit time to the region $1C2$ (negative when power is transferred *from* $1C2$).

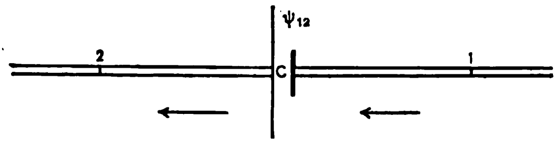


Fig. 72.

While the current I traverses the circuit, the conductor $1C2$ receives energy at the rate

$$P + P' = (\Psi_{12} + \Psi_{12}')I$$

and its resistance dissipates energy in heat at the rate

$$dH/dt = RI^2$$

Hence, by the principle of the conservation of energy,

$$(\Psi_{12} + \Psi_{12}')I = RI^2 \tag{41}$$

whence

$$\Psi_{12} + \Psi_{12}' = RI \qquad (a)$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (42)

$$I = (\Psi_{12} + \Psi_{12}')/R \qquad (b)$$

(42) (*a*) states that the impressed e.m.f. in any conductor is equal and opposite to the counter e.m.f. of resistance ($-RI$).

(42) (*b*) states that the current in any conductor is equal to the impressed e.m.f. acting upon the conductor divided by its resistance.

Either of these statements constitutes *Ohm's law* in its general integral form.

20. General Differential Form of Ohm's Law. Impressed Electric Intensity. The impressed e.m.f. in a region may be regarded

as the line integral of a *total* or *impressed* electric intensity $E =$ vector sum of e and E', the field intensity F/q, in the direction of the current density it produces. Thus

$$\Psi_{12} + \Psi_{12}' = \int E dL = \int (e + E')dL \tag{43}$$

along a line of impressed identity, $(e + E')$ being a vector sum.

Applying (41) and (43) to a stream-tube whose cross-section is dS at a point P where the impressed intensity is $E = e + E'$, we have, for the element of volume $d\tau = dL \cdot dS$ enclosing P and bounded by the sides of the tube and two right cross-sections distant dL apart,

$$E dL \cdot idS = (e + E')dL \cdot idS = ri^2 dLdS = i^2/k \cdot dLdS$$

whence

$$i = kE = k(e + E') = E/r = (e + E')/r \tag{44}$$

which is the general differential form of Ohm's law.

When $E = E'$, or $e = 0$, (44) reduces to (16).

23. Ohm's Law for Constant Current. Let the electrodes 1 and 2 of the conductor 1 C 2, Fig. 72, be maintained at the potential difference V_{12} (with the assistance of an intrinsic e.m.f. located outside 1 C 2, if necessary) while the conductor 1 C 2 is traversed

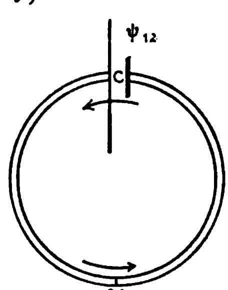


Fig. 73.

by a constant current I. Then electromagnetic energy is generated in the conductor at the rate $PI = \Psi_{12} I$, and electromagnetic energy is transferred from the field into the conductor at the rate $P' = \Psi_{12}' I = V_{12} I$, the total impressed e.m.f. in the conductor in the direction 1 C 2 being thus $V_{12} + \Psi_{12}$.

In this case, therefore, Ohm's law (42, *b*) becomes

$$I = (V_{12} + \Psi_{12})/R \tag{45}$$

(1) If $V_{12} = 0$, that is, if the electrodes 1 and 2 are connected together as in Fig. 73, to form a closed circuit, (45) becomes

$$I = \Psi_{12}/R \tag{46}$$

which is Ohm's law for a closed isolated circuit traversed by a constant current.

(2) If $\Psi_{12} = 0$, that is, if the conductor $1C2$ is homogeneous without an intrinsic e.m.f., (45) becomes

$$I = V_{12}/R \tag{47}$$

which is Ohm's law for a homogeneous conductor traversed by a constant current.

(3) If $V_{12} + \Psi_{12}$ is greater than zero, I has the direction $1C2$; if $V_{12} + \Psi_{12}$ is less than zero, I has the opposite direction.

(4) If $V_{12} + \Psi_{12} = 0$, that is, if $V_{12} = -\Psi_{12} = \Psi_{21}$, $I = 0$, and the agent with intrinsic e.m.f. Ψ_{12} is on *open circuit*. Thus the difference of potential between the terminals of a voltaic cell, or other agent possessing an intrinsic e.m.f. when no current is flowing, or when on open circuit, is equal in magnitude to the intrinsic e.m.f. for zero current, but has the opposite direction.

24. Mechanical Analogue of the Relation $V_{12} + \Psi_{12} = RI$, etc. Let an incompressible liquid flow at the constant rate I units

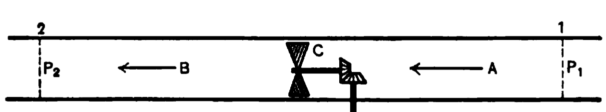


Fig. 74.

volume per second in the direction 12 of the arrows, Fig. 74, across every section of a pipe AB containing a screw propeller, or pump, C producing a difference of pressure Ψ_{12}, in the direction of the current ($1C2$), on its two sides. If the sections P_1 and P_2 are maintained at pressures V_1 and V_2, or at the pressure

difference $V_{12} = V_1 - V_2$, the total *fall* of pressure along the pipe from P_1 to P_2 is $V_{12} + \Psi_{12} = + RI$, and the rate at which work is done against friction within the volume P_1P_2 is $(V_{12} + \Psi_{12})I = RI^2$, R having the significance attached to it in § 18.

If the pipe is bent around and P_1 and P_2 joined together, so as to form a closed circuit, $V_1 = V_2$ or $V_{12} = 0$, and $\Psi_{12} = RI$.

If elastic membranes are stretched across the pipe at P_1 and P_2 (either before or after P_1 and P_2 are joined together), the propeller will force liquid from the region A into the region B (analogous to the positive and negative charges of the terminals of an agent possessing an intrinsic e.m.f., when on open circuit) until the pressure in B exceeds the pressure in A by the amount $V_{12} = \Psi_{12}$, numerically, when the current will cease.

25. The Fall of Potential Around a Closed Circuit. Consider a closed circuit containing an agent with an intrinsic e.m.f. Ψ and traversed by a constant current I. Let the resistance of the agent, called the *internal resistance*, be denoted by B, and that of the rest of the circuit, called the *external resistance*, by R, both conductors being supposed homogeneous. Then we have, by (46),

$$\Psi = BI + RI (= V_b + V_r) \tag{48}$$

Now BI denotes the fall of potential, V_b, in the direction of the current through the resistance B of the agent, and RI the fall of potential, V_r, in the direction of the current through the external resistance R. But the total fall of potential around a complete circuit is zero (§ 18, I.). Hence at the seat of the intrinsic e.m.f. there is a rise in potential in the direction of the current equal to $\Psi = (B + R)I$. The equation $V_1 - V_2 = \Psi_{12}$, § 23, (4), is a particular case of this proposition ($R =$ infinity).

To make the fall of potential as great as possible through the external circuit it is clear that R/B should be made as great as possible, if Ψ is independent of the current.

The e.m.f. of an agent is, in general, a more or less complicated function of the current, although there are some cases in

which the e.m.f. is constant for all values of the current. The limiting value which an e.m.f. approaches as the current approaches zero, and the resistance infinity, or the e.m.f. on open circuit, is, as we have seen, equal and opposite to the potential difference between its terminals on open circuit. If V denotes this potential difference, and if Ψ above is independent of the current, we have, for all values of the current,

$$V = V_b + V_r \tag{49}$$

26. Kirchhoff's Law II. In any closed circuit in a network of conductors traversed by steady currents, as the circuit $1\ 2\ 3\ 4 \cdots n$ in Fig. 75, the algebraic sum of all the intrinsic e.m.f.s is equal

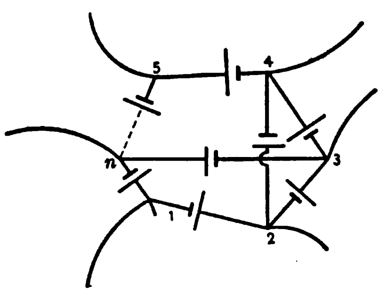


Fig. 75.

to the algebraic sum of the products RI. That is, if Ψ_{12}, Ψ_{23}, $\cdots$, Ψ_{n1} denote the intrinsic e.m.f.s in the branches 12, 23, $\cdots n1$, R_{12}, R_{23}, $\cdots$, R_{n1} the resistances of the same branches, and I_{12}, I_{23}, $\cdots$, I_{n1} the currents, both currents and e.m.f.s being reckoned positive in the same direction, as $12 \cdots n1$, around the circuit, then

$$\Sigma\Psi = \Sigma RI \tag{50}$$

For, by (45)

$$V_{12} + \Psi_{12} = R_{12}I_{12}$$
$$V_{23} + \Psi_{23} = R_{23}I_{23}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V_{n1} + \Psi_{n1} = R_{n1}I_{n1}$$

from which, by adding up both members separately, we obtain (50).

27. Wheatstone's Bridge consists of a network of six conductors arranged as in Fig. 76 or Fig. 77, with an intrinsic e.m.f. Ψ in the branch 13, Fig. 76, or the branch 24, Fig. 77. Let the currents in the branches be denoted by A, B, C, D, F, and G, as shown in the figure, the current in any branch being positive when in the direction of the arrow-head in that branch; and let the corresponding resistances of the branches be denoted by a,

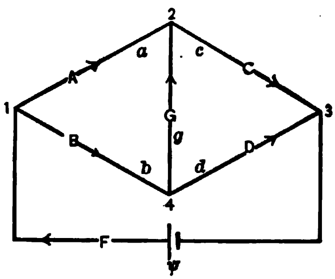


Fig. 76.

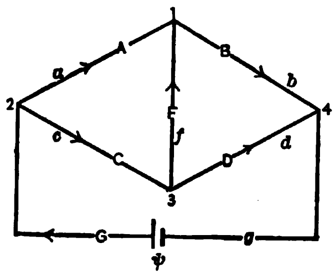


Fig. 77.

b, c, d, f, and g. First we shall find, from Kirchhoff's laws, the current G in the branch 24 when the e.m.f., Ψ, is in the branch 13, Fig. 76.

Applying Kirchhoff's law I. to each of the sets of conductors meeting in the points 1, 2, and 4, we obtain the relations

$$\left.\begin{aligned} F &= A + B \\ C &= A + G \\ D &= B - G \end{aligned}\right\} \qquad (a)$$

Applying Kirchhoff's law II. to each of the closed circuits 1241, 2342, and 1431, we obtain

$$-Gg + Aa - Bb = 0$$

$$Gg + Cc - Dd = 0$$

$$Ff + Bb + Dd = \Psi$$

On eliminating C, D, and F by (a), and rearranging, these equations become

$$-gG + aA - bB = 0$$

$$(g + c + d)G + cA - dB = 0$$

$$-dG + fA + (b + d + f)B = \Psi$$

from which

$$G = \Psi\,(bc - ad)/Q \tag{51}$$

if Q is written for the determinant of the coefficients of the currents in the last set of equations, viz.,

$$\begin{vmatrix} -g & a & -b \\ (g + c + d) & c & -d \\ -d & f & (b + d + f) \end{vmatrix} = Q \tag{52}$$

The current in any other branch can be found in the same manner.

The difference of potential between the points 4 and 2 is

$$V_{42} = Gg = \Psi(bc - ad)g/Q \tag{53}$$

and may be made as small a fraction of Ψ as desired by giving a suitable value to $(bc - ad)$.

From (41) and (52) and a comparison of Figs. 76 and 77, we see that when the e.m.f. Ψ is in the branch 24, Fig. 77, the current in the branch 13 is

$$F = \Psi\,(bc - ad)/Q \tag{54}$$

Thus the current in the conductor 24 due to e.m.f. in the conductor 13 is equal to the current in the conductor 13 due to the same e.m.f. in the conductor 24, all the resistances remaining unaltered.

By a similar method, this reciprocal relation may be shown to hold for any two branches of the network, or any network.

When $bc = ad$, the current in either of the two conductors 13 or 24, due to an e.m.f. in the other, is zero. The two conductors are then said to be *conjugate*. In this case the conductor in which there is no current may be removed, or its resistance

may be altered in any manner, without affecting the state of the rest of the system.

From the relation $I_{12} = (V_{12} + \Psi_{12})/R_{12}$ and the principle of superposition of potentials and e.m.f.s, it follows immediately that if any number of e.m.f.s is placed in the network, each will produce in any part of the system the same current it would have produced if acting alone. The current in any branch is thus the algebraic sum of the currents due to each e.m.f. separately.

Suppose an e.m.f. placed in one of the branches 12, 23, 34, or 41, Fig. 76. It will produce a current in the other branches, including 24. But if $bc = ad$, the e.m.f. in 13 will produce no current in 24, whatever this e.m.f. may be. Suppose the e.m.f. Ψ to have such a magnitude and direction as to produce a current in the branch 13 exactly neutralising the current in the same branch due to the e.m.f. in the other branch. Then there is no current in the branch 13, and it may be removed, or its resistance may be made infinite, without affecting the currents in the other branches. Thus, when $bc = ad$, the current in 24 is independent of the resistance, as well as of the electromotive force, in the branch 13. This result is applied below to Mance's method of measuring the resistance of a conductor containing an intrinsic e.m.f.

The condition that the two conductors 13 and 24 may be conjugate, viz., $bc = ad$, can be found very simply as follows. Let the voltages between the points 12, 23, 34, etc., Figs. 76 and 77, be denoted by V_{12}, V_{23}, V_{34}, etc. Then we have, as the condition that G may be zero in the arrangement of Fig. 76, $V_{24} = 0$. We have also, in this case, $A = C$, and $B = D$. Therefore

$$V_{12} = aA = V_{14} = bB$$

and

$$V_{24} = cA = V_{43} = dB$$

Dividing $aA = bB$ by $cA = dB$, we obtain $a / c = b / d$, or

$$bc = ad \tag{55}$$

In exactly the same way we find that only this same condition must be satisfied in order that F may vanish in the arrangement of Fig. 77.

Hence the conductors 13 and 24 are conjugate when this condition is satisfied.

The Wheatstone's bridge is very extensively applied to the comparison of electrical resistances.

28. The Comparison of Electrical Resistances. Thus suppose we have an unknown resistance a which is to be compared with a standard resistance c. The resistances a and c are connected up with two other resistances b and d, whose ratio must be known, as in Fig. 76 or Fig. 77, and the terminals of a battery, or other agent with an intrinsic e.m.f., are connected to the points 1, 3 or 2, 4, and an electrometer (or galvanometer, XII.) to the points 2, 4 or 1, 3. Then the resistance c, or the resistances b and d, or all three, are varied until the needle of the electrometer (or galvanometer, which is almost invariably used) remains undeflected whether the branch containing the battery is opened or closed. Then, by (55), $a = c\,b\,/\,d$.

29. Mance's Method of Determining the Resistance of an Agent with an Intrinsic E.M.F. The agent whose resistance, a, is to be determined is connected up as in Fig. 76 with three other resistances c, b, and d, at least one of which, together with the ratio of the other two, is known. A galvanometer or electrometer G is connected to the points 2, 4, and a wire containing a key, but no e.m.f., to the points 1, 3. Then the resistances c, b, and d, or at least one of them, are varied until the deflection of the galvanometer or electrometer is the same whether the key connecting the points 1 and 3 is open or closed. When this condition is reached, the current through the branch 24, or the voltage V_{42}, is independent of the resistance of the conductor 13, and the two conductors are conjugate. Hence

$$a = c\,b\,/\,d$$

30. Kelvin's Method of Measuring the Resistance of a Galvanometer or other Current Indicator. The galvanometer or electrometer of § 28 is removed and is replaced by a wire W containing a key, and the instrument whose resistance is to be determined is put in place of the unknown resistance a. Then one or more of the resistances, b, c, d, are varied until, the battery circuit being closed, the permanent indication of the instrument (a deflection, if a galvanometer or similar instrument is under experiment; silence, if a telephone) remains constant when the key in the wire W is opened or closed. Then no current traverses the wire in either case, and $a = cb/d$, as in § 28.

31. Kelvin's Double Bridge furnishes the most accurate known means of comparing two very small resistances.* The conductor A, Fig. 76, of a Wheatstone's bridge arranged for the comparison of resistances, § 28, is disconnected from the conductor B at the point 1, and the terminals of one of the resistances to be compared, x, are connected to the free end of A, denoted by 1′, and the original point 1. In like manner, C is separated from D and the other resistance under comparison, y, is connected to the point 3 and the free end of C, denoted by 3′. The bridge is completed by joining the points 1′ and 3′ with a third conductor of low resistance. The resistances a, b, c, d (all, in practise, of considerable magnitudes, which can be determined with precision by other methods), or either c and d or a and b, are then varied, the ratio b/d being kept constantly equal to a/c, until no current traverses the galvanometer. Then, since the currents through a and c are equal, and also the currents through b and d, and therefore the currents through x and y, it is clear that

$$x/y = a/c = b/d \tag{56}$$

For a thorough discussion of the Kelvin bridge in its general form, see *Zeitschr. für Instrumentenkunde*, Feb. and Mar., 1903. Equation (56) does not express the general condition for a balance.

*With the possible exception of the shunted differential galvanometer method (F. Kohlrausch, *Wied. Ann.*, Vol. 20, p. 76, 1883).

CHAPTER IX.

ELECTROLYTIC AND METALLIC CONDUCTION.

1. Metallic Conduction. The electric current in a metallic conductor, whether a pure metal or an alloy, in the solid or liquid state, is not, so far as is known, associated with any chemical change in the conductor or with the convection of its *molecules* or *atoms* from one part to another. All substances which conduct in this manner are said to conduct *metallically*. A theory of metallic conduction, based on the motion of *electrons*, will be referred to in §15.

2. Electrolytic Conduction. Electrolysis. Ions. The electric current in most chemical compounds, however, is invariably associated with their separation into two constituents, atoms or groups of atoms, called *ions*. These ions do not appear separately in the body of the conductor, but only at the electrodes by which the current enters and leaves it. Hence one of the ions moves toward the anode, and is therefore called the *anion;* while the other moves toward the kathode, and is called the *kation.*

Substances in which the electric current is associated with the transportation of atoms or molecules are called *electrolytes*, the process of electro-separation of the constituents is called *electrolysis*, and the substances are said to conduct *electrolytically.*

The simplest electrolytes, in some respects, are molten salts, *e. g.*, KCl at a temperature above 734° C. During the electrolysis of this salt, K appears at the kathode and Cl at the anode. Thus K is the kation and Cl the anion. As in this case, so in the electrolysis of salts, acids, and bases generally, the kation invariably consists of a metal or hydrogen, and the

anion of an acid element or radical (sometimes combined with a metal).

The commonest electrolytes are aqueous and other solutions of salts, acids, and bases. In such a solution, all together an electrolyte, the dissolved substance, not the solvent, is separated into the moving ions. Hence the dissolved substance itself is often spoken of as the electrolyte. It must be remarked, however, that pure dry acids, salts, and bases at ordinary temperatures, as well as pure water and other solvents whose solutions are frequently electrolytes, are either not conductors, or else possess extremely small conductivities.

The actual determination of the constituents forming the ions is sometimes a matter of considerable difficulty. For in many cases the ions do not themselves separate out at the electrodes, but on reaching the electrodes, combine with them chemically, if such reaction is possible, or with the solvent, if the first reaction is impossible. If neither reaction is possible, the ions collect at the electrodes or are there liberated.

Thus if molten KCl is electrolysed with a platinum anode and a kathode of graphite, metallic potassium may be collected at the kathode, but at the anode the Cl unites with the platinum.

Also, if an aqueous solution of H_2SO_4 is electrolysed between platinum electrodes, H (the kation) appears at the kathode, where it is given off as a gas (no reaction with platinum or with water being possible); and SO_4 (the anion) appears at the anode. But SO_4 can neither combine with platinum, nor can it exist alone in the presence of water; hence it reacts with the latter to form H_2SO_4 and O, the first going into solution, and the second being evolved as a gas.

3. The Laws of Faraday. Electrolytic Measurement of Current. According to the experiments of Faraday, confirmed by all later investigation,

I. The mass of an ion deposited on an electrode, or there dissolved, during the passage of a current, is proportional to the

electric charge crossing the electrode during the deposit or solution. Thus, if M_a denotes the mass of an ion a deposited at an electrode, or there dissolved, in the time t, while the charge q crosses the electrode (or passes through the electrolyte).

$$M_a = K_a q \tag{1}$$

where $K_a = M_a / q$ is a constant for the ion a, equal to the mass of the ion deposited per unit charge, and called its *electrochemical equivalent.*

If a condenser of capacity S is repeatedly charged to a voltage V and discharged in a constant direction through an electrolyte at the rate n times per second, the mass M_a of the ion a deposited or dissolved in the time t will be

$$M_a = K_a q = K_a SVnt \tag{2}$$

from which, by measuring M_a, S, V, n, and t, K_a may be determined. If M_a is expressed in grams, t in seconds, and S and V in *RES* units, K_a, will be expressed in the *RES unit electrochemical equivalent.*

If a constant current I traverses the electrolyte, we have

$$M_a = K_a q = K_a It \tag{3}$$

from which

$$I = M_a / K_a t \tag{4}$$

Hence by measuring M_a, t and K_a, I may be determined. If M_a and t are measured in grams and seconds, respectively, and K_a in the *RES* unit, I will be expressed in the *RES unit current.*

II. The electrochemical equivalent of any ion is directly proportional to its atomic (or combining) weight and inversely proportional to its valence; *i. e.*, the electrochemical equivalent of a substance is proportional to its chemical equivalent. Thus if a and b denote two ions, A and B their atomic (or combining) weights, a' and b' their valences, and K_a and K_b their electrochemical equivalents, then

$$K_a/K_b = \frac{A/a'}{B/b'} \tag{5}$$

whence

$$K_a = A/a' \cdot b'/B \cdot K_b \tag{6}$$

If therefore the combining weights and valences of all ions are known, and the electrochemical equivalent of any one of them, the electrochemical equivalents of all the rest may be found from (6). The ion whose electrochemical equivalent has been most accurately determined is the silver ion, of which the valence is 1, the combining (atomic) weight 107.93 ($O = 16.000$), and the electrochemical equivalent 0.001119 gram/coulomb (XIV.). Hence if b in (6) denotes silver, the electrochemical equivalent of any other ion a is

$$\begin{aligned} K_a &= A/a' \cdot 1/107.93 \cdot \times 0.001119 \text{ gram/coulomb} \\ &= A/a' \cdot \times 0.00001037 \text{ gram/coulomb} \end{aligned} \tag{7}$$

By omission, law II. states that K_a is independent of the nature of the compound in which a is found, the nature of the solvent if the compound is in solution, the strength of the current, the temperature, and other physical conditions.

If an element has two or more valences in different compounds, then it exists as two or more distinct ions, each with its own electrochemical equivalent. Thus iron in ferric compounds, as $FeCl_3$, has a valence 3, while in ferrous compounds, as $FeCl_2$, its valence is 2. Hence

$$\text{K (ferric iron)/K (ferrous iron)} = \frac{\text{valence ferrous iron}}{\text{valence ferric iron}} = \frac{2}{3}$$

since the atomic weight of iron is the same for both classes of compounds.

Since the same quantity of electric charge crosses every section of a conductor carrying a steady current in any given time, it follows from law II. that in the electrolysis of any compound the ions are deposited simultaneously at the two electrodes in the same proportions in which they occur in the compound. Thus

when KCl is electrolysed, for every atom of K deposited at the kathode, an atom of Cl is deposited at the same time at the anode. Likewise, in the electrolysis of an aqueous solution of H_2SO_4, for every ion SO_4 deposited at the anode, two atoms of H are deposited at the same time upon the kathode. Since one sulphion reacts with a molecule of water to form a molecule of H_2SO_4 and an atom of O, it follows that H and O are evolved at the kathode and anode respectively in the proportions in which they occur in water, while the total quantity of H_2SO_4 in solution remains constant. In like manner, when an aqueous solution of silver nitrate is electrolysed between silver electrodes, for every silver atom (kation) deposited upon the kathode, one nitrion NO_3 (anion) is liberated at the silver anode, and reacts with an atom of this electrode to form a molecule of silver nitrate. The nitrate goes into solution. Thus the total quantity of salt and the total quantity of silver in solution remain constant, while the kathode gains as much silver as the anode loses.

4. The Arrhenius Theory of Electrolytic Dissociation. According to this theory, which, though not universally accepted, serves to explain many electrochemical phenomena, the molecules of a molten salt, or a salt in aqueous solution, or other electrolyte, are always, in greater or less numbes, independently of the passage of a current, broken up, or *dissociated*, each into two kinds of atoms or atomic groups, called *ions*. One kind of ion is positively charged, and is called a *kation*, the other is negatively charged and is called an *anion*. Thus a molecule of H_2SO_4 dissociates into two positively charged H kations and one negatively charged SO_4 anion; and a molecule of KCl into a negative chlorine anion and a positive potassium kation, the metallic or hydrogen atoms being in general the kations, and the acid atoms or radicals the anions.

Every ion of the same valence carries a charge of the same magnitude, and this charge is directly proportional to the valence of the ion. Thus the negative charge of a chlorine ion is

equal to the positive charge of a potassium ion, the valence of each being 1; the negative charge of a sulphion, whose valence is 2, is equal to the positive charge of two hydrogen ions, whose valence is 1; the charge of a silver ion is equal to the charge of a nitrion, and to the charge of a hydrogen, potassium, or chlorine ion; and so on.

The electric conduction current, or transfer of electric charge, through an electrolyte consists in a convection current of the ions — the kations with their positive charges moving toward the (negative) kathode, and the negative anions moving toward the (positive) anode. On reaching an electode the ions give up their charges to the electrodes, and react with one another, or with the solvent, or with the electrode, to form molecules.

If we assume that the current *at* each electrode consists in the motion of only *one* kind of ion, which will presently be proved to be true, the theory affords a simple explanation of the laws of Faraday:

Since a given kind of ion always carries an electric charge of the same magnitude and sign, the quantity of an ion deposited upon an electrode will be proportional to the charge which crosses the electrode. This is Faraday's first law.

Since the charge of an ion is proportional to its valence, and since the mass of an ion is proportional to its combining weight, the mass of ions of one kind carrying a given charge, or the ion mass deposited per given charge, must be proportional to its combining weight and inversely proportional to its valence. This is the second law of Faraday.

5. A Gram Atom of an element is a quantity of the element equal, in grams, to the number denoting its atomic weight. Thus a gram atom of potassium is 39.15 grams potassium.

A Gram Ion is in the same way a quantity of the ion equal, in grams, to the number denoting its combining weight. Thus a gram ion of silver is 107.93 grams silver, and a gram sulphion is $96.06 = (32.06 + 4 \times 16.00)$ grams SO_4.

A gram molecule of a substance is a quantity of the substance equal, in grams, to the number denoting its molecular weight. Thus the molecular weight of KCl is $39.15 + 35.45 = 74.60$, and therefore 74.60 grams KCl is a gram molecule of this substance.

It will now be obvious that the magnitude of the charge carried by a gram ion of every univalent ion is the same. The charge carried by a gram ion of a bivalent ion is twice as great, and so on. The charge carried by a gram ion is equal, numerically, to the gram ion divided by the mass, in grams, carrying unit charge, or equal to the gram ion divided by the electrochemical equivalent of the ion. Thus the silver gram ion is 107.93 grams, and the electrochemical equivalent of silver is 0.001119 gram per coulomb. Hence the charge carried by a gram ion of silver *or any other univalent ion*, which will be denoted by Q, is

$$Q = 107.93 \text{ grams} \div 0.001119 \text{ gram/coulomb} \\ = 96450 \text{ coulombs} \tag{8}$$

The concentration of a solution is the quantity of dissolved substance per unit volume of solution. The concentration may be expressed in grams / c.c., grams / liter, gram molecules / liter, etc., and will be denoted here by C.

6. Velocities of the Ions. Hittorf's Ratio. Hittorf's Numbers, etc. If in the electrolysis of a solution ordinary convection and diffusion effects are prevented, it is found that no change takes place in the concentration of the solution except in the vicinity of the electrodes. Owing to the deposit or liberation of the ions at the electrodes, however, the total quantity of dissolved substance diminishes (reactions at the electrodes which produce the substance being neglected). Hence it follows that the concentration of the dissolved substance, or rather, the concentration of the solution, diminishes near the electrodes (the reactions mentioned being neglected, if occurring).

To study the matter more closely and to make the conditions perfectly definite, consider the electrolysis of an aqueous solution of silver nitrate between platinum electrodes, A and K. If we imagine a porous partition P, preventing diffusion and convection currents across it but not hindering the motion of the ions, placed between A and K, but not close to either, the quantity of $AgNO_3$ in each of the compartments into which P divides the electrolytic vessel will diminish during electrolysis. Before electrolysis begins, each compartment contains as many anions as kations ($+$ Ag and $-$ NO_3), one of each being necessary to form a molecule of silver nitrate. During electrolysis let K be the kathode and A the anode.

Let the velocities of the kations and anions in the main body of the solution, at the partition for example, be denoted by U and V respectively, and let $U/V = h$. Then for every h kations which cross the partition in the direction AK, 1 anion crosses in the direction KA. During this process the number of molecules of $AgNO_3$ in the compartment KP is diminished by h, since both ions are necessary to form a molecule; and likewise the number of molecules of the salt in the compartment AP is diminished by 1. Hence (*Hittorf's law*)

$$\text{Loss of salt near anode}/\text{Loss of salt near kathode} = U/V = h \quad (9)$$

The ratio $h = U/V$ is called *Hittorf's ratio*, and will be further discussed below.

When h kations have crossed the partition in the direction AK, and one anion therefore in the direction KA, $h+1$ anions are left without the corresponding kations at the anode A, and $h+1$ kations are left without the corresponding anions at the kathode K. The free kations immediately give up their positive charges to the kathode, and are deposited upon it; and the free anions immediately give up their negative charges to the anode, and react with water to form nitric acid and oxygen.

Since each ion carries the same numerical charge, the electric current in the main body of the electrolyte is the same as if all

the anions were moving with the velocity $U+V$ toward A, and all the kations were at rest; or as if all the kations were moving with the same velocity $U+V$ toward K, and all the anions were at rest.

Since, moreover, by what precedes, $h+1$ kations give up their charges to the kathode, and $h+1$ anions give up their charges to the anode, while only h kations cross the partition in the direction AK and only 1 anion in the direction KA, and since the current is the same across every section of the conductor, the current *at* the anode consists in the motion of anions only, and the current *at* the kathode consists in the motion of kations only. It does not follow, however, that the velocity of either anion or kation at an electrode is $U+V$. For the quantity of kations deposited at the kathode is greater than the quantity crossing the partition toward the kathode in the ratio $(h+1)/h=(U+V)/U$; and the quantity of anions deposited on the anode is greater than the quantity crossing the partition toward the anode in the same time in the ratio $(h+1)/1=(U+V)/V$.

Thus a fraction $U/(U+V)$ of the total quantity of the kation deposited at the kathode in any interval comes from the main body of the electrolyte, and a fraction $1-U/(U+V)=V/(U+V)$ comes from the vicinity of the kathode. In like manner, a fraction $V/(U+V)$ of the total quantity of the anion deposited at the anode in any interval comes from the main body of the electrolyte, and a fraction $1-V/(U+V)=U/(U+V)$ comes from the vicinity of the anode.

The velocity, U_1, of the kations at or very near the kathode, and the velocity, V_1, of the anions at the anode, can be obtained from the condition that the charge crossing every section of the conductor in the same interval is the same. This condition gives, for the interval in which N anions, and therefore hN kations, cross the partition,

$$U_1 hN(h+1)/h=hNU+NV$$

whence

$$U_1=(U^2+V^2)/(U+V)=[(h^2+1)/(h+1)]V$$
$$=[(h^2+1)/h(h+1)]U$$

and

$$V_1 N(h + 1) = NV + hNU$$

whence

$$V_1 = (U^2 + V^2) / (U + V) = U_1 \tag{10}$$

The ratios $U/(U+V)$ and $V/(U+V)$, which represent the fractions of the total current carried in the main body of the electrolyte by the kations and anions respectively, are called the *transport numbers* of the kation and anion, respectively, for the given electrolyte. Put

$$U/(U+V) = n \tag{11}$$

then

$$V/(U+V) = 1 - n \tag{12}$$

and

$$U/V = h = n/(1-n) \tag{13}$$

If the electrolysis of silver is carried on between silver electrodes instead of platinum electrodes, the loss of the salt around the kathode will be the same, for a given charge, as before; but since the total quantity of salt in solution now remains constant, there will be a *gain* of salt around the anode equal to the loss around the kathode. If from the amount of $AgNO_3$ corresponding to the total quantity of silver deposited on the kathode (or dissolved at the anode), which would be the gain at the anode if the silver ions did not move, we subtract the actual gain at the anode, we obtain the loss of salt at the anode due to the motion of the silver ions. The actual gain in salt at the anode (equal to the loss at the kathode) divided by this quantity is Hittorf's ratio, from which n and $1 - n$ are easily computed. Since any quantity of the salt is proportional to the quantity of silver in the salt, we may use the quantities of silver deposited on the kathode and gained by the solution around the anode instead of the corresponding quantities of salt, and much more conveniently. Also, the loss at the kathode instead of the gain at the anode may be obtained, if preferable, by direct experiment.

Hittorf's ratio, and therefore the transport numbers n and $1 - n$, are found to vary slightly with the temperature, h and n increas-

ing as the temperature rises. Thus, for NaCl, $n = 0.392$ at 20° C. and 0.449 at 95° C.; and for $AgNO_3$, $n = 0.470$ at 10° C. and 0.490 at 90° C.

For certain electrolytes, especially aqueous solutions of alkaline salts, n is almost independent of the concentration. Thus, in the case of an aqueous solution of KCl, n changes from 0.497 to 0.486 when the concentration increases from 0.03 to 2.5 grm. mol./lit.

For other electrolytes n decreases rapidly with the concentration. Thus, for an aqueous solution of $CuSO_4$ n decreases from 0.36 to 0.27 as the concentration increases from 0.1 to 2.0 grm. mol. per liter.

For still other electrolytes n increases rapidly with the increase of concentration. Thus, for an aqueous solution of $AgNO_3$, n changes from 0.474 to 0.53 as the concentration increases from 0.01 to 2 grm. mol. / liter.

In all cases n is independent of the current strength.

7. The Dissociation Ratio. By several methods, into a discussion of which we cannot here enter, for example the lowering of the freezing point produced by the solution of a substance, the ratio of the number of dissociated molecules in a solution to the total number of molecules of the dissolved substance can, according to the modern dissociation theory, be determined without the use of an electric current. This ratio, called the *dissociation ratio*, will be denoted by m. For a given solution m in general increases slowly with the temperature.

If the concentration of a given kind of electrolytic solution is diminished, m in general increases, very rapidly at first, then more and more slowly, reaching, when the solution becomes very dilute, sensibly the constant value 1. That is, in very dilute solutions all the molecules of a dissolved electrolytic substance are dissociated.

8. Ohm's Law for a Homogeneous Electrolyte. Conductivity and Molecular Conductivity. Since according to the dissociation

theory the electric current in an electrolyte consists in the motion of the ions, the undissociated molecules playing no part in the conduction, the current density, i, must be equal to the continued product of the concentration of the salt, C, in gram molecules per cc., the dissociation ratio, m, the valence of the ions, a', the quantity of charge carried by a univalent gram ion, Q, and the sum of the ionic velocities, $U+V$. For mC is the number of gram ions of each kind per cc., $mCa'Q$ is the charge upon all the kations in one cc., and, numerically, the charge upon all the anions in one cc.; hence $mCa'QU$ and $mCa'QV$ are the total positive and negative charges, respectively, crossing unit area per second in opposite directions. Hence

$$i = mCa'Q(U+V) \qquad (14)$$

Now Ohm's law holds rigorously for liquid electrolytes (though not, in general, for gases), as well as for metallic conductors. Hence the above equation may be written

$$i = kE = mCa'Q(U+V) \qquad (15)$$

Therefore the sum of the velocities of the ions, $U+V$, is proportional to the electric intensity E. Hence, since n or h is independent of the electric intensity or current (§6), the velocity of each ion must be proportional to E. If therefore u and v denote the velocities of the kation and anion respectively per unit intensity, we have, when the intensity is E

$$U = uE \quad \text{and} \quad V = vE \qquad (16)$$

(15) may therefore be written

$$i = kE = mCa'Q(u+v)E \qquad (17)$$

so that

$$k = mCa'Q(u+v) \qquad (18)$$

The ratio of the conductivity k to the concentration C (in grm. mol./cc.) is called the *molecular conductivity*, and will be denoted by M. Thus

$$M = k/C = ma'Q(u+v) \qquad (19)$$

When the solution becomes very dilute, m becomes sensibly equal to unity; hence, if the corresponding values of M, k, C, u, and v are denoted by these letters with the subscript zero, we have from (19)

$$M_0 = k_0 / C_0 = a' Q(u_0 + v_0) \tag{20}$$

From the last two equations

$$m = M / M_0 \cdot (u_0 + v_0) / (u + v) \tag{21}$$

The conductivity, k, of an electrolyte is readily measured by methods similar to those used in the case of metallic conductors, except that to avoid the troublesome effects of polarisation an alternating current and a telephone or electrodynamometer are employed instead of a direct current and a galvanometer. From the conductivity and the concentration the molecular conductivity is obtained by division from (19). From the molecular conductivity M, at given dilution, and M_0, m can be computed, if

$$(u_0 + v_0) / (u + v)$$

is known.

9. Variation of Electrolytic Conductivity with Temperature Pressure, and Viscosity. The conductivity of an electrolytic solution always increases rapidly with the temperature. Since in equation (18) a' and Q are constants, and mC does not vary much with the temperature, this increase in conductivity must be almost wholly due to an increase in $(u + v)$. Such an increase in the velocities would be expected from the fact that the viscosity of a liquid rapidly diminishes with the increase of temperature.

The conductivity of an electrolytic solution also increases with the pressure to which it is subjected. Since this increase occurs in the case of very dilute solutions ($m = 1$), although to a less degree than for strong solutions, it must be due, in part at least, to the diminutions of the liquid's viscosity (which always diminishes with increase of pressure) and the consequent increase of the ionic velocities.

For solutions in which different solvents contain the same number of gram ions of a given substance per cc., the conductivity is always less the greater the viscosity of the solvent, the conductivity of the solvent being eliminated if appreciable.

The intimate connection between the ionic velocities and viscosity is shown further by the fact that the conductivities of analogous compounds in solution bear to one another the same ratios as do their velocities of diffusion.

10. Molecular Conductivity and Concentration. The Dissociation Ratio. The relation between the molecular conductivity M of an aqueous solution of KCl and the dilution of the solution, expressed in terms of the number of liters of water containing a

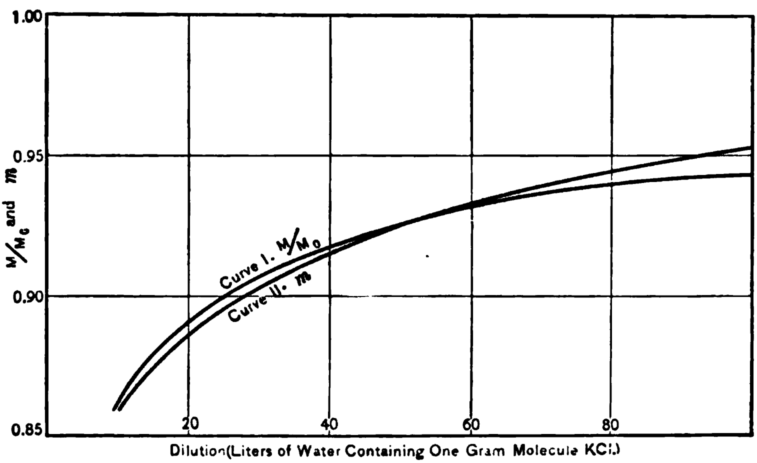


Fig. 78.

gram molecule of the salt, is shown graphically in Fig. 78. As the dilution increases, or the concentration diminishes, M increases, rapidly at first, then more and more slowly, reaching a constant value M_0 when the solution becomes very dilute.

Curve *I* of Fig. 78 shows the relation between M/M_0 and the dilution, and curve *II* the relation between the dissociation ratio m, calculated from the lowering of the freezing point, and the dilution, for the same salt KCl in aqueous solution.

The close similarity of the curves shows that, very approximately, in this case

$$m = M/M_0 \tag{22}$$

Hence, to the same degree of approximation, equation (21) gives $(u_0 + v_0) = (u + v)$; that is, the sum of the velocities of the ions is nearly independent of the concentration. In practise this approximation is frequently employed, and the dissociation ratio is calculated from (22) on the assumption that $(u_0 + v_0) = (u + v)$ sensibly. Equation (22), however, appears in many cases to be not even approximately true except for very great dilutions.

11. The Velocities of the Ions. The Law of Kohlrausch. From (19) it follows that

$$(u + v) = M/ma'Q \tag{23}$$

and from (9) and (16) that

$$h = U/V = uE/vE = u/v \tag{24}$$

From these equations we have

$$u = hM/ma'Q \text{ cm. per second per unit intensity} \tag{25}$$

and

$$v = (1 - h)\,M/ma'Q \text{ cm. per second per unit intensity} \tag{26}$$

from which u and v can be readily computed, all the quantities in the second members being capable of experimental determination.

The ionic velocities, when calculated by these formulæ, for extremely dilute solutions are found to be wholly independent of the compounds in which they occur. This is the *law of Kohlrausch.* Thus the velocity of the chlorine ion is the same whether in a very dilute solution of HCl or a very dilute solution of NaCl. The velocities of the hydrogen, silver, hydroxyl (OH), and chlorine ions, all in cms. per second, when the electric intensity is one volt per cm., are 0.00320, 0.00057, 0.00181, and 0.00069, respectively, by calculation from (25) and (26).

The above results have been confirmed by the *direct* experiments of Lodge and of Whetham (*Philosophical Transactions*, A, 1893, and A, 1895).

12. Thermal Analogue of Ohm's Law. Let the two parallel faces, 1 and 2, distant L apart, of a large plane slab of a homogeneous isotropic substance with thermal conductivity k' be maintained at the temperatures t_1 and t_2. Near the center of the slab the flow of heat from 1 to 2 is perpendicular to the faces, and the fall of temperature per unit length, or temperature gradient, from 1 to 2 is uniform and equal to $(t_1 - t_2)/L = E'$. The time rate per unit area, i', at which heat crosses a plane surface within the slab parallel to its faces is

$$i' = k' E'$$

which is strictly analogous to Ohm's law.

13. The Variation of Metallic Conductivity with Temperature. The resistivity of all pure metals increases with the temperature, the relation between the resistivity and temperature being approximately linear for ordinary temperatures according to the equation

$$r_t = r_{t_0}[1 + a(t - t_0)]$$

The coefficient a is called the *resistance temperature coefficient.* For many metals a is approximately equal to $1/273$, the temperature coefficient of the expansion of a gas at constant pressure, when t_0 is taken as 0° C. or 273° absolute. For such a substance we have approximately from the above formula

$$r_t = t/273 \cdot r_{273}$$

where t is the absolute temperature at which the resistivity is r_t, and r_{273} is the resistivity at the temperature 273° absolute.

14. The Law of Wiedemann and Franz. Not only is the law of Ohm analogous to the law of thermal conductivity, but for nearly all metals and alloys the ratio of the thermal conductivity k' to the electrical conductivity k is approximately the same at a

given temperature, and is proportional to the absolute temperature. This indicates that the processes involved in the two kinds of conduction are largely identical.

15. The Electron Theory of Conduction. According to the comprehensive and rapidly developing *electron* theory, an atom is constituted of a multitude of minute particles, called electrons, each carrying a permanent and constituent electric charge, whose magnitude is that of the charge carried by a univalent ion in the electrolysis of liquids. In a neutral atom, the number of positive particles is equal to the number of negative particles; in a charged atom or radical, the number of positive electrons exceeds the number of negative electrons, or *vice versa*, by 1, 2, 3, etc., according as the ion is univalent, bivalent, trivalent, etc. The charge of a single electron is the smallest electric charge which can exist, and no charges exist except the charges of electrons.

In the electrolytic conduction of liquids no free electrons take part, but the current consists, as we have already seen according to the dissociation theory, in the convection of the ions, all of atomic magnitude.

In electric conduction through gases, which is also electrolytic, the positive ions are atomic in magnitude (atoms or radicals plus or minus one or more electrons of the same sign), but the negative ions are frequently single electrons, though either may be loaded down with an agglomeration of neutral molecules. The electric current consists in the convection in opposite directions of these ions. The kathode rays, emitted from the kathode in a highly exhausted vacuum tube, consist of negative electrons only, moving with velocities of the same order as that of light.

In metallic conduction the main body of the metal does not participate in the conduction. The current consists in the convection of (temporarily) free positive electrons in the direction of the current, or in the convection of (temporarily) free nega-

tive electrons in the opposite direction, or in both processes simultaneously. The number of electrons taking part in metallic conduction is small in comparison with the number taking part in electrolytic conduction proper, and the velocity is relatively great.

Thermal conduction also takes place by the motion of electrons, but under a temperature gradient both positive and negative electrons move in the same direction.

For an extended discussion of the electron theory as applied to electric and thermal conduction in metals, reference must be made to memoirs by Drude, *Ann. der Physik*, I., p. 566, 1900, III., p. 369, 1900, and VII., p. 687, 1902. For a general treatment of the electron theory, see also Lord Kelvin, *Philosophical Magazine* (6), III., p. 257, 1902, and Larmor's *Æther and Matter.* For a sketch of the development of the electron idea, with abundant references, see Kaufmann, *Physikalische Zeitschrift*, III., p. 9, 1901, or a translation of the same in *The Electrician*, November 8, 1901. An elementary treatment of the subject is given by Lodge in *The Electrician*, Vols. 50 and 51, 1902–1903.

CHAPTER X.

THERMAL AND VOLTAIC ELECTROMOTIVE FORCES.

1. The Law of Volta. Around a circuit made up of any number of different metals connected end to end, there is no resultant e.m.f., and therefore no electric current, when all parts of the circuit are at the same temperature (unless there is a changing magnetic flux through the circuit, XIII.).

2. The Seebeck Effect. Around a circuit formed of two different metals a current flows, in general, when the two junctions are at different temperatures.

Such a circuit is called a *thermocouple*, or a *thermoelement.*

In a thermocouple consisting of a copper wire and an iron wire, if the mean temperatute of the junctions is less than 275° C., a current flows from the copper to the iron across the hot junction.

3. The Peltier Effect. When an electric current flows across the junction of two metals, heat is there, in general, either absorbed (thermal energy transformed into electrical energy) or emitted (electrical energy transformed into heat), according to the direction of the current. The rate of the energy transformation is proportional to the current strength, and the process is completely reversible.

Thus at a copper-iron junction heat is absorbed when the current passes from Cu to Fe; and heat is emitted at the same rate when the same current crosses the junction (maintained at the same temperature) in the opposite direction.

The energy transformations occurring during the circulation of the current in the thermoelement of copper and iron, § 2, thus tend to cool the hot junction and to heat the cold junction.

The Peltier E.M.F. The junction of two metals is therefore the seat of an intrinsic e.m.f., called the *Peltier e.m.f.*, which is constant at a given temperature of the junction for all values of the current. The e.m.f. varies with the nature of the metals and with the temperature of the junction. At the junction of two metals A and B, at the temperature t, the Peltier e.m.f. acting from A to B will be denoted by ${}_tP_{ab}$. See § 1, I.

If the two junctions of a thermoelement have the same temperature t, ${}_tP_{ab}$, the e.m.f. from A to B, will have the same value at both junctions. Hence no current will traverse the circuit, but a difference of potential will be developed, B coming to a potential ${}_tP_{ab}$ higher than that of A.

Since, however, ${}_tP_{ab}$ is a function of the temperature, there will, when the two junctions are at different temperatures t_1 and t_2, be a resultant Peltier e.m.f. around the circuit, equal, when measured in the direction around the circuit from A to B across the junction at temperature t_2, to

$$ {}_{t_2}P_{ab} + {}_{t_1}P_{ba} = {}_{t_2}P_{ab} - {}_{t_1}P_{ab} \tag{1} $$

4. The Thomson Effect. When an electric current traverses a conductor along which there is a temperature gradient, heat is either absorbed (transformed into electrical energy) or emitted (electrical energy transformed into heat), according to the direction of the current, throughout the portion of the conductor in which the temperature gradient exists. The rate of energy transformation is directly proportional to the strength of the current, and depends, so far as temperature is concerned, only on the temperatures of the ends of the conductor. The energy transformations are perfectly reversible, changing sign, but not magnitude, with the direction of the current.

The absorption or emission of heat just described takes place in addition to the evolution of heat according to Joule's law at a rate proportional to the square of the current.

The Thomson E.M.F. A conductor in which there is a temperature gradient is therefore the seat of an intrinsic e.m.f. which

is constant for all values of the current for given temperatures of its terminals. This e.m.f. is called the *Thomson e.m.f.* in the conductor, and is considered positive when it is directed from the lower to the higher temperature. If t_1 and t_2 denote the temperatures of the cooler and hotter ends of a conductor A, the Thomson e.m.f. from the cooler to the hotter end is denoted by ${}_{t_1t_2}T_a$.

The fact that the Thomson e.m.f. (or the corresponding energy transformations) depend, so far as temperature is concerned, only on the temperatures of the ends of the conductor, follows from the *law of Magnus*, which states that in a circuit composed of a single homogeneous metal there is no electric current, howsoever the temperature varies from point to point. Thus the Thomson e.m.f. from a point A to another point B of the circuit is the same either way around the circuit. (The law of Magnus, and the deduction therefrom, do not hold in certain extreme cases, as when the cross-section of the conductor changes suddenly, etc.; also, at least in certain cases, when a portion of the circuit is magnetised, when it is, strictly, non-homogeneous.)

Let $IS_a dt$ denote the rate of heat absorption in the elementary length dL of a conductor A, the mean temperature within the length dL being t and the rise in temperature from one end to the other being dt, when the current I flows up the temperature gradient. Then the Thomson e.m.f. up the temperature gradient dt along dL is

$$dT_a = S_a dt \tag{2}$$

S is, in general, a function of the temperature and varies from substance to substance. If the temperatures of the cooler and hotter ends of the conductor are t_1 and t_2, respectively, we have, on integrating (2) from one end of the conductor to the other,

$${}_{t_1t_2}T_a = \int dT_a = \int_{t_1}^{t_2} S_a dt \tag{3}$$

By an obvious thermal analogy, S_a is called the *specific heat of electricity* in the metal A at the temperature t.

In certain substances, *e. g.* copper, S is positive; that is, heat is absorbed when the current flows up the temperature gradient. In others, as iron, S is negative; that is, heat is absorbed when the current flows down the temperature gradient.

In a copper-iron thermoelement, therefore, with junctions at temperatures t_1 and t_2, the Thomson e.m.f. in each metal has the same direction as the current in that metal — up the gradient in the copper, and down the gradient in the iron.

If S_a and S_b denote the value of S at the temperature t for two metals A and B forming a thermocouple with the cold and hot junctions at temperatures t_1 and t_2, respectively, the total Thomson e.m.f. around the circuit in the direction from A to B across the hot junction is

$$_{t_1t_2}T_a + {}_{t_2t_1}T_b = {}_{t_1t_2}T_a - {}_{t_1t_2}T_b = \int_{t_1}^{t_2}(S_a - S_b)dt \qquad (4)$$

5. The Total Thermal Electromotive Force in a Circuit consisting of two homogeneous metals is the sum of the two Peltier e.m.f.s at the junctions and the two Thomson e.m.f.s along the conductors. Thus if Ψ_{ab} denotes the total e.m.f. in the circuit, measured in the direction around the circuit from A to B across the hot junction,

$$\Psi_{ab} = {}_{t_2}P_{ab} - {}_{t_1}P_{ab} + \int_{t_1}^{t_2}(S_a - S_b)dt \qquad (5)$$

6. The Law of Intermediate Metals (Becquerel's Law I.). If at one of the junctions, at temperature t, of two metals A and B forming a thermoelement a third metal C is inserted between A and B, and if the two resulting junctions are kept at the original temperature t of the junction AB before the insertion of C, the total e.m.f. of the circuit is not altered. The total Thomson e.m.f. in C is evidently zero, since its two ends are at the same temperature; hence the law states that

$$_tP_{ab} = {}_tP_{ac} + {}_tP_{cb} \qquad (6)$$

Thus two wires may be soldered together, instead of being welded or twisted, without affecting the e.m.f. of the junction.

§§ 3, 4, and 6 completely account for the law of Volta.

7. The Law of Successive Temperatures (Becquerel's Law II.). Consider a thermoelement of two metals A and B with the junctions at temperatures t_1 and t_2, respectively. Let the total e.m.f. around the circuit in the direction from A to B across the junction at temperature t_2 be denoted by ${}_{t_1t_2}\Psi_{ab}$. Then, with similar nomenclature, if experiments are made with one junction of the element at temperature t_1 and the other at t', then with the first at temperature t' and the other at t'', and so on, and finally with the first at temperature t_n and the other at temperature t_2, it will be found that, whatever the values of n, t', t'', $\cdots$, t_n,

$$ {}_{t_1t_2}\Psi_{ab} = {}_{t_1t'}\Psi_{ab} + {}_{t't''}\Psi_{ab} + \cdots + {}_{t_nt_2}\Psi_{ab} \tag{7} $$

8. Thermoelectric Power. If $d\Psi_{ab}$ denotes the total thermal electromotive force around the circuit of a thermoelement the direction from A to B across the junction whose temperature is t, when the temperature of the other junction is $t + dt$, the differential coefficient $d\Psi_{ab}/dt$, the e.m.f. per unit difference of temperature, is called the *thermoelectric power* of the metal A with respect to the metal B at the temperature t, or the *thermoelectric power* of the thermoelement AB at the temperature t, and will be denoted be ${}_tp_{ab}$. Thus we have

$$ {}_tp_{ab} = d\Psi_{ab}/dt \tag{8} $$

For the total e.m.f. in the circuit in terms of the thermoelectric power p, we have from (7) and (8)

$$ {}_{t_1t_2}\Psi_{ab} = \int d\Psi_{ab} = \int_{t_1}^{t_2} {}_tp_{ab}dt \tag{9} $$

9. The Thermoelectric E.M.F. of a Copper-Iron Element at Moderate Temperatures. The relation between the total thermal electromotive force of a copper iron thermoelement and the temperature t of one of the junctions, when the other junction is kept at the constant temperature of 0° C., is shown graphically in Fig. 79, for temperatures up to 600° C. As t increases, the e.m.f. around the circuit in the direction from copper to iron at

the junction whose temperature is t increases from a negative value, for t less than 0° C., to a maximum positive value when $t = 275°$ C. As t continues to increase, the e.m.f. decreases, falling to zero at $t = 550°$ C. Beyond this temperature the e.m.f. is negative, as when t was less than 0° C., that is, the current flows (or the e.m.f. is directed) from iron to copper across the junction at temperature t.

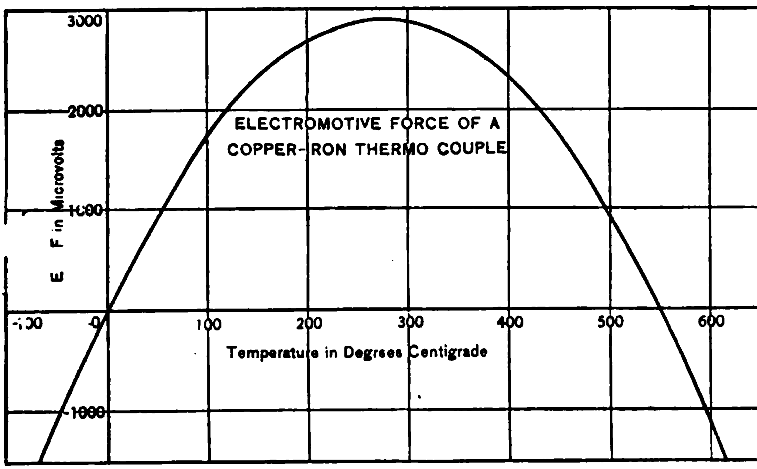


Fig. 79.

From the curve in Fig. 79, the thermoelectric power of copper with respect to iron, ${}_tp_{ci}$, can be readily obtained for all temperatures within the limits of the curve. For if a tangent is drawn from the point of the curve corresponding to any temperature t, the tangent of the angle made by this line with the axis of temperatures is

$$d\Psi_{ci}/dt = {}_tp_{ci}$$

The relation between ${}_tp_{ci}$ and t is given in the curve of Fig. 80, which is a straight line. That is, the thermoelectric power of copper with respect to iron is a linear function of the temperature, or

$$ {}_tp_{ci} = d\Psi_{ci}/dt = A_1 t + A_2 \qquad (10)$$

where A_1 and A_2 are constants for the given element (copper-iron).

The temperature corresponding to the point $B(t = 275°$ C.), for which ${}_t p_{ci} = 0$, is called the *neutral temperature* for copper and iron. The temperature corresponding to the point C ($t = 550°$ C.), in passing which the electromotive force around the circuit is reversed in direction when one of the junctions is at 0° C., is called the *temperature of inversion* of copper and iron with respect to the temperature 0° C. of the cooler junction.

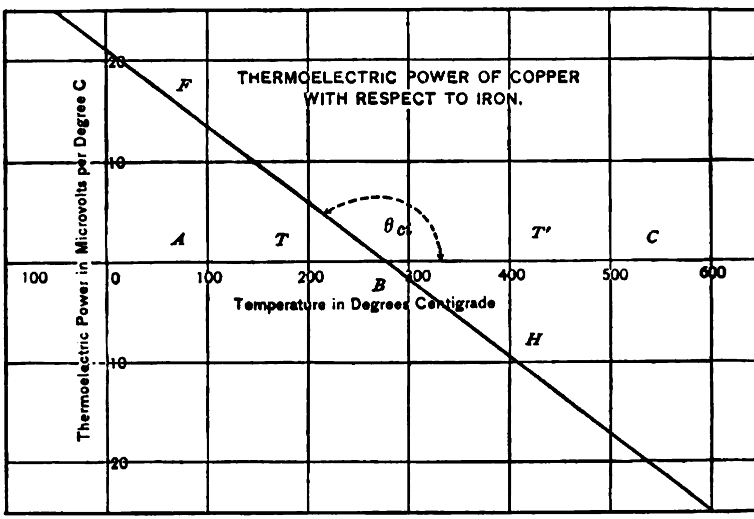


Fig. 80.

The total e.m.f. Ψ_{ci} of the couple is represented in this figure by the area included between the curve and the axis of temperatures, and between the two perpendiculars dropped from the curve to the points on this axis corresponding to the temperatures of the junctions. Thus, if the junctions are at temperatures T and A,

$$ {}_{AT}\Psi_{ci} = \text{area } ATFA $$

In like manner,

$$ {}_{AB}\Psi_{ci} = \text{area } ABFA $$

$$ {}_{AT'}\Psi_{ci} = \text{area } ABFA + \text{(negative) area } BT'HB $$

etc. This follows directly from (9).

The curve showing the relation between the thermoelectric power of an element and the temperature is called the *thermoelectric line* of that element. The thermoelectric lines of nearly all thermoelements consisting of either pure metals or alloys are straight lines over a considerable range of temperature, like that of the copper-iron couple, Fig. 80.

If the thermoelectric line of a given element is a straight line, the curve showing the relation between the total thermal electromotive force in the circuit and the temperature t of one junction, the temperature t_1 of the other being kept constant, is a parabola. For we have, by integration of (10) between the limits t_1 and t,

$$ {}_{t_1t}\Psi_{ct} = \int_{t_1}^{t} d\Psi_{ct}/dt\,dt = A_1(t^2 - t_1^2) + A_2(t - t_1) \qquad (11) $$

which is the equation of a parabola with its axis in the negative direction of the axis of e.m.f.s.

10. Becquerel's Law III. At a given temperature the thermoelectric power of a metal A with respect to a metal C is equal to

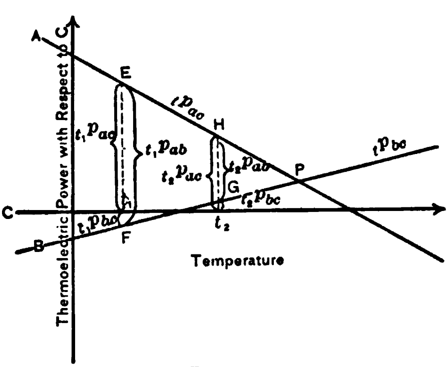


Fig. 81.

the thermoelectric power of the metal A with respect to any other metal B plus the thermoelectric power of B with respect to C. That is,

$$ {}_tp_{ac} = {}_tp_{ab} + {}_tp_{bc} \qquad (12) $$

Hence if the thermoelectric lines are drawn for two metals A and B with respect to the same metal C (Fig. 81), ${}_tp_{ab}$ at any

temperature t can be obtained by subtracting from the ordinate ${}_tp_{ac}$ the ordinate ${}_tp_{bc}$.

The total e.m.f. ${}_{t_1t_2}\Psi_{ab}$ is given by the area *EFGHE* of the figure.

P, the point of intersection of the two lines A and B, is the *neutral point*, or the point corresponding to the neutral temperature, for the metals A and B, since there

$${}_tp_{ab} = {}_tp_{ac} - {}_tp_{bc} = 0$$

11. The Thermoelectric Circuit Treated as a Reversible Thermodynamic Engine. So far as the Thomson and Peltier effects are concerned, the absorption and evolution of heat in a thermoelectric circuit are, as we have seen, proportional to the current strength and the time, or to the total charge which has passed through the circuit, and completely reversible, changing sign with the direction of the current. There are other thermal processes going on in the circuit, however, which are not reversible: the conduction of the heat from the hotter to the cooler junction, which bears no direct relation to the electrical phenomena; the evolution of heat according to Joule's law at a rate proportional to the square of the current; and the radiation of heat, which, like its conduction, bears no direct relation to the electrical phenomena. Since by diminishing the current the second effect, being proportional to the square of the current, may be made as small as we please in comparison with the Thomson and Peltier effects, which are proportional to the first power of the currents; and since the first and third effects have no direct relation to the electrical phenomena; we shall assume that the total Thomson and Peltier e.m.f.s are not affected by these irreversible processes, and that the relations between them can be obtained by treating the circuit as a perfectly reversible thermodynamic engine, all irreversible effects being neglected. The application in this manner of the principles of thermodynamics to the matter in question is justified by the approximate agreement with experiment of the results to which it leads.

The first law of thermodynamics, or the principle of the conservation of energy, together with experiment, has furnished us with the relation

$$ {}_{t_1t_2}\Psi_{ab} = {}_{t_2}P_{ab} - {}_{t_1}P_{ab} + \int_{t_1}^{t_2}(S_a - S_b)dt \qquad (13) $$

If we apply (13) to the case in which $t_1 = t$, and $t_2 = t + dt$, or if we simply differentiate (13) with respect to t, we obtain

$$ d\Psi_{ab} / dt = d({}_tP_{ab}) / dt + S_a - S_b \qquad (14) $$

The second law of thermodynamics furnishes another relation. Let dH denote the quantity of heat *absorbed* into the circuit at the temperature t, while a charge dq (= current × time) is carried around the circuit once. Then we have, for the whole cycle, by the second law of thermodynamics,

$$ 0 = \int dH/t = dq\left[{}_{t_2}P_{ab}/t_2 - {}_{t_1}P_{ab}/t_1 + \int_{t_1}^{t_2}(S_a - S_b)/t\,dt\right] $$

or

$$ {}_{t_2}P_{ab}/t_2 - {}_{t_1}P_{ab}/t_1 + \int_{t_1}^{t_2}(S_a - S_b)/t\,dt = 0 \qquad (15) $$

the temperature being expressed on the absolute scale.

Applying this equation to the case in which $t_1 = t$ and $t_2 = t + dt$, or simply differentiating the equation with respect to t, we obtain

$$ d({}_tP_{ab}/t)/dt + (S_a - S_b)/t = 0 \qquad (16) $$

(16) may be written

$$ d({}_tP_{ab})/dt - {}_tP_{ab}/t + S_a - S_b = 0 \qquad (17) $$

The combination of this equation with (14) gives

$$ {}_tp_{ab} (= d\Psi_{ab}/dt) = {}_tP_{ab}/t \qquad (18) $$

or

$$ {}_tP_{ab} = {}_tp_{ab}t = t\,d\Psi_{ab}/dt \qquad (19) $$

The combination of (16) and (19) gives

$$ S_a - S_b = -t\,d({}_tp_{ab})/dt = -t\,d({}_tP_{ab}/t)/dt = -t\,d^2(\Psi_{ab})/dt \qquad (20) $$

Since at the neutral temperature for the metals A and B ${}_tp_{ab} = 0$, (19) gives, for this temperature,

$$ {}_tP_{ab} = t_t p_{ab} = 0 \tag{21} $$

That is, when one junction of two metals is at the neutral temperature, there is at this junction no Peltier e.m.f. and no absorption or evolution of heat, t, the absolute temperature, being always greater than zero.

If θ_{ab} denotes the angle made with the axis of temperatures by the tangent to the thermoelectric line of A with respect to B at the point on the line corresponding to the temperature t, then (18) gives

$$ \tan\theta_{ab} = d({}_tp_{ab})/dt = d({}_tP_{ab}/t)/dt \tag{22} $$

(20) and (22) give

$$ S_a - S_b = -t\tan\theta_{ab} \tag{23} $$

In the common case in which ${}_tp_{ab}$ is a linear function of the temperature, or the thermoelectric line straight, $\tan\theta_{ab}$ is constant (K_{ab}) for all temperatures; and (23) becomes

$$ S_a - S_b = -K_{ab}t \tag{24} $$

If θ_{ab} is greater than 90°, $\tan\theta_{ab} = K_{ab}$ is negative, and $S_a - S_b$ therefore positive. For copper-iron (Fig. 80) $S_a - S_b$ is thus always positive.

The experiments of Le Roux and of Tait have shown that for lead and for certain platinum-iridium alloys S is excessively small or zero. Hence denoting the metal lead by L, and putting $S_l = 0$, we have

$$ S_a = S_a - S_l = -td({}_tp_{al})/dt = -td({}_tP_{al}/t)dt = -t\tan\theta_{al} $$

Hence, as a matter of convenience, lead is chosen as a standard metal, and the thermoelectric lines of all other metals and alloys are, in general, drawn with respect to this metal.

12. The Thermoelectric Diagram. In Fig. 83 are drawn the thermoelectric lines with respect to lead of a number of metals and alloys. The line for lead of course coincides with the axis of

temperatures, and all other substances for which $S = 0$ have lines parallel to this axis. The system of thermoelectric lines is known as the *thermoelectric diagram.*

As an introduction to the use of the thermoelectric diagram, we shall consider in detail the ideal thermoelectric lines A and B of two metals A and B with respect to lead, A being a straight line, as in the common case, and B an irregular curve, Fig. 82. The construction of the remainder of the figure is sufficiently

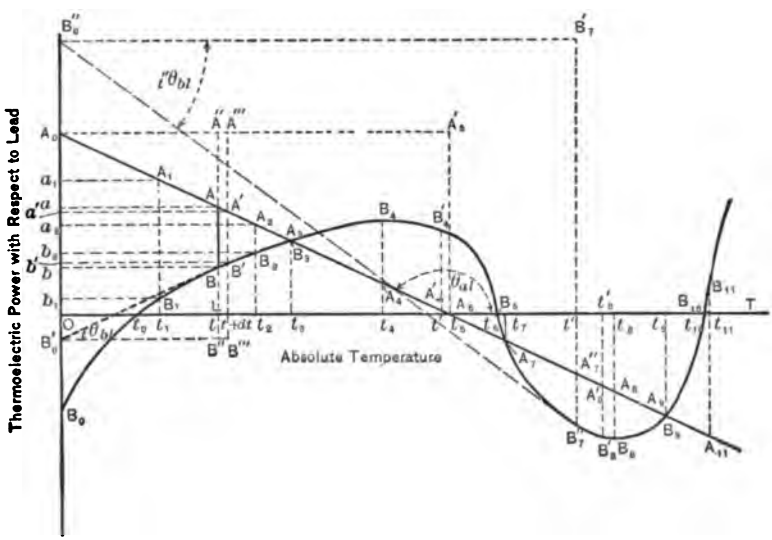


Fig. 82.

obvious, all the lines being either parallel or perpendicular to the axis of temperatures, or tangential to the line B.

The thermoelectric power of A with respect to lead, ${}_tp_{al}$, is OA_0 at the absolute temperature zero. From this value it regularly decreases as a linear function of the temperature, passing through the value LA at the temperature t, and reaching 0 at the temperature t_5. This point is the neutral temperature of A with respect to lead. Beyond t_5, ${}_tp_{al}$ is negative.

The thermoelectric power of B with respect to lead, ${}_tp_{bl}$, is OB_0, a negative quantity, at 0° absolute. At t_0, a neutral tem-

perature of B with respect to lead, ${}_t p_{bl} = 0$. Between t_0 and t_6, ${}_t p_{bl}$ is positive, having a maximum value at t_4. At t_6 and t_{10}, other neutral temperatures of B with respect to lead, ${}_t p_{bl}$ is again zero, reaching between them, at the temperature t_8, a negative maximum.

At the temperature t, ${}_t P_{al} = t\,{}_t p_{al} = aA \times AL =$ area $ALOaA$. When $t = 0$, this area is $O \times OA_0 = 0$; when $t = t_5$, its value is $OA_5 \times O = 0$; beyond t_5, $ALOaA$ is below the line OT, or is negative. Thus at t_9, ${}_{t_9}P_{al} = t_9\,{}_{t_9}p_{al} = Ot_9 \times t_9A_9$, the thermoelectric power t_9A_9 being negative, and the area lying wholly below OT.

Similarly, at t^0, ${}_t P_{bl} = t\,{}_t p_{bl} =$ area $BLObB$. When $t = 0$, this area is $O \times OB_0 = 0$; between $t = 0$ and $t = t_0$, the area is negative, or below OT; at t_0, t_6, and t_{10}, the neutral temperatures for B and lead, the area is zero; from t_0 to t_6 the area is positive, and from t_6 to t_{10} negative.

At t^0, ${}_t S_a = t \times (-\tan {}_t\theta_{al}) = A_0A'' \times AA''/A_0A'' = AA''$. Since $\tan {}_t\theta_{al}$ is constant and negative (θ_{al} greater than $90°$ and less than $180°$), $S_a = AA''$ is always positive, or AA'' is always drawn upward from A.

The quantity ${}_t S_a\,dt$ is equal to $AA'' \times A''A''' =$ area $Aaa'A'A$.

In like manner, at t^0, ${}_t S_b = t \times (-\tan {}_t\theta_{bl}) = B_0'B'' \times BB''/B_0'B'' = BB''$. When the point B'' is below the point B (${}_t\theta_{bl}$ less than $90°$), $S_b = BB''$ is negative. Thus from $t = 0$ to $t = t_4$, and from $t = t_7$ to $t = t_{10}$, S_b is negative; while from $t = t_4$ to $t = t_8$, S_b is positive. At the temperature $t = t''$ it has the positive value $B_7''B_7'$.

The quantity $-\,{}_t S_b\;dt$ is equal to area $BB'B'''B''B =$ area $Bbb'B'B$.

At the temperature t, ${}_t p_{ab} = {}_t p_{al} - {}_t p_{bl} = LA - LB = BA$. BA is positive, or A is above B, from $t = 0$ to $t = t_3$ (a neutral temperature for the thermocouple AB), and from $t = t_7$ to $t = t_9$ (additional neutral temperatures for the couple AB), but is negative (A below B) from $t = t_3$ to $t = t_7$.

The total thermal electromotive force in a circuit consisting of the two metals A and B, with junctions at temperatures t_1 and t_2, is

$$_{t_1t_2}\Psi_{ab} = \int_{t_1}^{t_2} {}_t p_{ab} dt = \int_{t_1}^{t_2} BAdt = \text{area } A_1B_1B_2A_2A_1$$

This result can be obtained also from the relation (13). For

$$(S_a - S_b)dt = \text{area } Aaa'A'A + \text{area } Bbb'B'B$$

$$\int_{t_1}^{t_2}(S_a - S_b)dt = \text{area } A_1a_1a_2A_2A_1 \ (1) \ + \text{area } B_1b_1b_2B_2B_1 \cdot (2)$$

and

$$_{t_2}P_{ab} - {}_{t_1}P_{ab} = \text{area } A_2a_2b_2B_2A_2 \ (3) \ - \text{area } A_1a_1b_1B_1A_1 \ (4)$$

Hence

$$_{t_1t_2}\Psi_{ab} = (1) + (2) + (3) - (4) = \text{area } A_1B_1B_2A_2A_1$$

as proved otherwise above.

If one junction of the thermoelement AB is kept at the constant temperature t_1, while the temperature t of the other junction, at first equal to t_1, is gradually increased, $_{t_1t}\Psi_{ab}$ will increase from zero, its value when $t = t_1$, until $t = t_3$, when it has the value $_{t_1t_3}\Psi_{ab} =$ area $A_1A_3B_1A_1$. If t is increased beyond t_3, to t' for instance, the e.m.f. will diminish, since

$$_{t_1t'}\Psi_{ab} = A_1A_3B_1A_1 + \int_{t_1}^{t'} {}_t p_{ab} dt = A_1A_3B_1A_1 \ (1)$$
$$- A_3B_4B_4'A_4'A_4A_3 \ (2)$$

$_t p_{ab}$ being negative between $t = t_3$ and $t = t_7$. When area (2) becomes equal in magnitude to area (1), $_{t_1t'}\Psi_{ab}$ is zero, and t' is a temperature of inversion for A and B with respect to t_1. As t is still further increased, the e.m.f. increases negatively until $t = t_7$, beyond which it increases algebraically, or decreases negatively, with another inversion at B_8' (area $B_8'A_8'A_7B_8'$ = area $A_7B_4'A_4'A_7$), until $t = t_9$. Beyond this temperature $_{t_1t}\Psi_{ab}$ increases negatively, inverting again at B_{11}, and thereafter remaining negative.

When $t_1 = t_3$ and $t = t_7$, the Peltier e.m.f.s at both junctions are zero, no heat being there absorbed or evolved, and ${}_{t_3t_7}\Psi_{ab} = {}_{t_3t_7}\Psi_{ab} =$ (negative) area $A_3B_4A_7A_4A_3$ (1) $= \int_{t_3}^{t_7}(S_a - S_b)dt.$ The

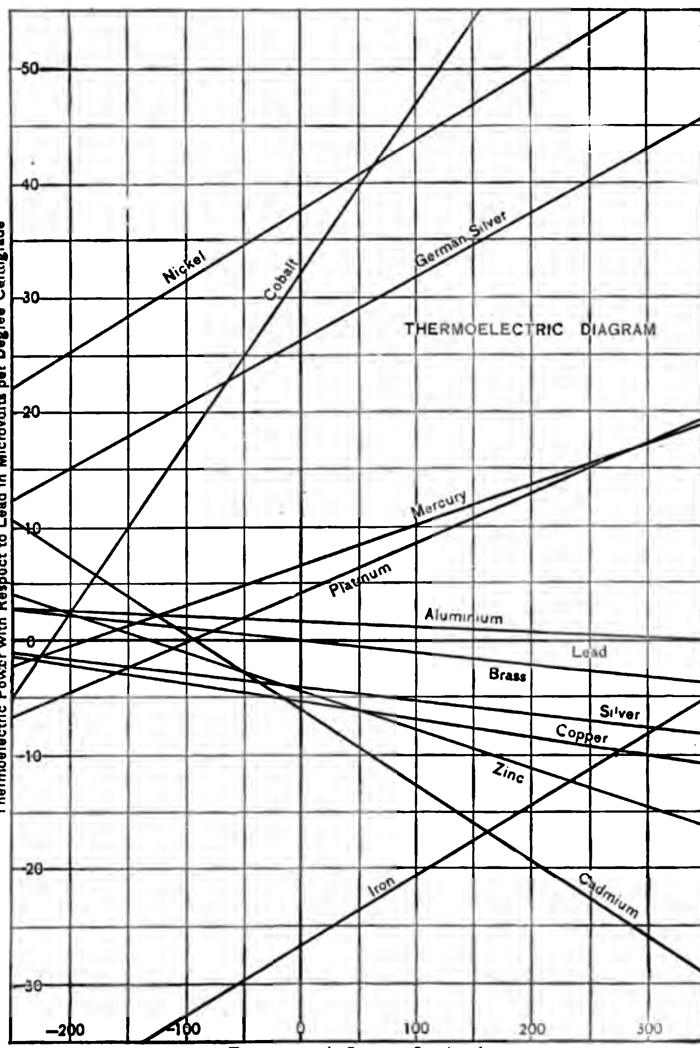


Fig. 83.

e.m.f. is negative, or the current flows (if the circuit is closed) from B to A across the hot junction at temperature t_7.

When $t_1 = t_7$ and $t = t_9$, ${}_{t_7t_9}\Psi_{ab} = {}_{t_7t_9}T_{ab} =$ area $A_7A_9B_7''A_7$ (2), and is positive, or directed from A to B across the hot junction.

If $t_1 = t_3$ and $t = t_9$, the Thomson e.m.f.s are still the only e.m.f.s in the circuit and ${}_{t_3t_9}\Psi_{ab} = (1) - (2)$. Since (1) is greater than (2), the resultant e.m.f. is negative, or the current flows from B to A across the hot junction.

For an extended discussion of the electron theory of thermal electromotive forces reference must be made to a previously mentioned article by Drude, *Ann. der Physik*, III., p. 369, 1900.

13. The Intrinsic E.M.F. of a Reversible Voltaic Cell. The Theory of Kelvin and von Helmholtz. By a *reversible* cell is meant a cell in which all the processes, both chemical and physical, are completely reversed when the direction of the current is reversed, the Joulean evolution of heat excepted. Such, for example, is a Daniell cell, which consists of a zinc electrode immersed in a solution of zinc sulphate and a copper electrode immersed in a solution of copper sulphate, all contained in the same vessel, interdiffusion of the two solutions being prevented by a porous cup between them or by other means. When a charge Q (§ 5, IX.) passes through the cell from the zinc ($z' = 2$) to the copper ($c' = 2$), one half gram ion of zinc goes into solution and one half gram ion of copper is deposited on the copper electrode; and when the same charge is passed through the cell in the opposite direction, one half gram ion of copper goes into solution and one half gram ion of zinc is deposited on the zinc electrode. The thermoelectric processes occurring at the contacts of the dissimilar substances are also reversible with the current. The Joulean heat, which is proportional to the square of the current and irreversible, may be made as small as desired in comparison with the energy transformed reversibly, which is proportional to the first power of the current, by diminishing the strength of the current.

Let the e.m.f. of a reversible cell at the absolute temperature t be denoted by Ψ. Let the electrodes be connected up to the plates of a continuously adjustable condenser so that by gradually diminishing or increasing the capacity a charge may be sent very slowly in either direction through the cell, the Joulean heat being made negligible.

Let the system now be carried through a reversible cycle as follows: (1) With the cell at the temperature t, let the capacity of the condenser be slowly increased until a small charge Q/n, where n is a large number, has passed through the cell. The voltage of the cell remaining constant through the process, external work will be done by the cell equal to $\Psi Q/n$. The energy of the condenser is increased by $\frac{1}{2}\Psi Q/n$, and the mechanical energy of the system increases by the same amount (§ 55, I.).

The source of the energy transformed by the cell is, in general partly chemical and partly thermal. (The law of Volta for a metallic circuit at uniform temperature does not hold for a circuit partly electrolytic.) Let J denote the net energy transformed from chemical into electrical energy when a charge Q traverses the cell at the temperature t in the direction of the e.m.f. Then, if J is not equal to ΨQ, an amount of heat

$$H = \Psi Q/n - J/n$$

is absorbed by the cell during the above process, according to the principle of the conservation of energy.

(2) Let the cell be cooled to the temperature $t - dt$. During the process an amount of thermal energy which may be made wholly negligible by sufficiently diminishing dt, is abstracted from the cell.

(3) With the cell at the temperature $t + dt$, let the capacity of the condenser be diminished until a charge Q/n has passed through the cell in the opposite direction. Then, if H' denotes the quantity of heat abstracted from the cell during this process,

$$H' = Q/n \cdot (\Psi - d\Psi/dt\, dt) - 1/n \cdot (J - dJ/dt\, dt)$$

(4) Finally, let the cell be heated to the original temperature t, a negligible quantity of heat, sensibly equal to that abstracted in (2), being absorbed. The cycle is now complete.

Applying the second law of thermodynamics to the cycle, we have

$$(H - H')/H = dt/t$$

that is

$$\{\Psi Q - J + [Q(\Psi - d\Psi/dt\,dt) - (J - dJ/dt\,dt)]/(\Psi Q - J)\} = dt/t$$

or

$$\Psi Q - J = Qt\,d\Psi/dt + t\,dJ/dt \tag{27}$$

which, since dJ/dt is, according to experiment, sensibly zero, may be written

$$\Psi Q - J = Q\,t\,d\Psi/dt \tag{28}$$

If $d\Psi/dt$ is positive, ΨQ is greater than J, or the work done by the cell is greater than the energy supplied by the chemical reactions, and a quantity of heat $H = \Psi Q - J$ is absorbed by the cell and transformed to make up for the deficiency. If $d\Psi/dt$ is negative, heat is given out by the cell. If $d\Psi/dt$ is zero, which is nearly true in the case of a Daniell cell, $\Psi Q = J$, and no thermal energy is on the whole transformed.

(28) may be written

$$\Psi = J/Q + t\,d\Psi/dt \tag{29}$$

which is von Helmholz's formula. From this formula the e.m.f. of a reversible cell can be calculated after observing J, Q, t, and $d\Psi/dt$. The agreement between the e.m.f. calculated in this manner and the e.m.f. determined by direct experiment is, in many cases, very close.

14. Intrinsic E.M.F. at a Single Interface. Single Difference of Potential. The formula of von Helmholz, just developed for a complete electrolytic cell, which may contain several electrolytes and always contains two electrodes, with an intrinsic e.m.f. at each interface (and sometimes throughout each electrolyte) can

obviously be applied also to the intrinsic e.m.f. at any one of the interfaces. For example, Ψ may denote the intrinsic e.m.f. acting from zinc to zinc sulphate in a Daniell cell, J the heat developed (if the energy is not used electrically) by the passage of one-half gram ion of zinc from the solid state into zinc ions, and $d\Psi/dt$ the temperature rate of change of the e.m.f. Then

$$\Psi = J/Q + t \cdot d\Psi/dt$$

The actual e.m.f. of the complete cell is equal to the algebraic sum of all such included e.m.f.s, and the actual temperature coefficient to the algebraic sum of all the individual temperature coefficients.

The intrinsic e.m.f. acting from the zinc (or other substance) to the electrolyte will develop a potential difference equal to Ψ in magnitude but directed from the electrolyte to the zinc (or other substance). Such a potential difference is called a *single difference of potential.* There is no satisfactory method of determining experimentally such a difference of potential, nor can it be computed from the above equation, since J can not be determined for any one kind of ions, as zinc ions, alone. For when one kind of ions goes into solution, another kind goes out of solution.

(29) is seen to include (19) as a particular case.

For additional information on the theory of the voltaic cell, single potential differences, etc., reference must be made to treatises on electrochemistry, where also references to the original literature may be found.

CHAPTER XI.

MAGNETS. MAGNETOSTATIC FIELDS.

1. Magnets. A bar of steel placed in a long helix of wire traversed by an electric current is found, on removal from the helix, to have acquired certain properties *analogous*, in many respects, to those of an electret, and is said to have been *magnetised* or to have become a *magnet*. The same name is applied to any body possessing these properties, however acquired, some of which will be described in the following pages.

2. Electric and Magnetic Analogues. Just as an electret and the region outside it are the seat of electric induction or displacement, so a magnet and the region around it possess *magnetic induction*. To the electrisation of the one corresponds the *magnetisation* of the other. *Tubes of induction* run through the magnet, entering at one end and issuing at the other, being *continuous* like an electret's tubes of displacement. There is, however, no magnetic analogue of a true electric charge, or discontinuity of electric displacement. *Magnetic conduction* and *magnetic conductors* do not exist, a tube of induction cannot be cut in two thus developing true magnetic charges, hence the induction is always continuous or the tubes of induction are always closed. Analogous to the electric intensity is the *magnetic intensity*, connected with the induction by a relation similar to that which connects electric intensity and displacement. Substances differ magnetically, or possess different *inductivities*, just as they possess different permittivities. Discontinuities in the magnetic intensity occur where the tubes of induction pass from one substance to another of different inductivity or through a substance whose inductivity continuously changes. Where these discon-

tinuities of intensity exist are *apparent magnetic charges*, or *quantities of magnetism*, analogous to apparent electric charges, arising from discontinuities in the electric intensity. The regions or surfaces in which the discontinuities occur are called the *poles* of the magnet. If the magnet is long and slender and the induction within it uniform and in the direction of its length, the poles are approximately concentrated at its ends. In every case, however, poles are more or less *distributed.*

Upon the poles of a magnet in a magnetic field, as upon the poles of an electret in an electric field, forces are found to exist, and from these forces and their seats the strengths of the poles are defined and their distribution determined, greater or less forces corresponding, in general, to greater or less pole strengths, and more or less restricted seats of the forces to more or less concentrated poles.

3. Positive and Negative Magnetic Poles. That pole of a magnet toward which, while in the magnetising helix, a right-handed screw placed with its axis coincident with that of the helix would have to be translated in order to rotate in the direction of the current around the helix is called the *positive* pole of the magnet, and the other pole the *negative* pole. The terms positive and negative in this connection are purely conventional, but are justified as in electrostatics (§ 1, I.).

4. The Earth's Magnetic Meridian at a Point. The Axis of a Magnet. North and South Magnetic Poles. A magnet so suspended or otherwise supported near the earth as to have perfect freedom of motion about its center of gravity will always take up a position with a definite line connecting the positive and negative poles, or rather a definite direction in the magnet, pointing in a definite geographical direction. This direction is, in general, *northerly* and *southerly*, and the vertical plane through it is called the *earth's magnetic meridian* at the point. The positive and negative poles point in the northerly and the southerly direction, respectively, and are therefore called *north* and *south*

poles, respectively. The definite direction in the magnet, from the negative to the positive pole, is called the *axis* of the magnet. This term is also applied to a certain *line*, with this direction, in the magnet (§ 24).

5. The Force Between Two Magnetic Poles. Between two magnetic poles a force always exists, repulsive or positive if the two poles have the same sign, and attractive or negative if the poles have opposite signs.

This force between two magnetic poles can be measured by any kind of a dynamometer provided that the magnets are so long and their poles so nearly concentrated at the ends that the poles under experiment are sensibly outside the field of influence of the other poles.

The force decreases rapidly with the increase of the distance between the poles.

At a given distance apart the force between the poles depends upon the medium in which they are immersed.

At a given distance apart and in a given medium the force is different for different pairs of poles.

6. Coulomb's Law of Force for Concentrated Poles of Permanent Magnets in an Infinite Homogeneous Isotropic Medium. Consider three approximately concentrated poles A, B, C, belonging to three very long, very slender, cylindrical magnets of very hard steel magnetised in a slender solenoid or helix, longer than the magnets, traversed by an electric current. The more nearly the poles are concentrated, the greater the ratio of the length to the diameter of each magnet, the greater the coercive intensity of the steel (§ 25, XIII.), and the more nearly homogeneous and isotropic the surrounding medium, the more nearly are the following relations found by experiment to be true:

The force between any two of the poles varies inversely as the square of the distance between them.

The force between two of the poles, as A and B, at any distance d apart in a given medium 1 bears to the force between the

same poles at the same distance d apart in another medium, 2, the same ratio (μ_2/μ_1, see below) which the force between another pair, as A and C, at any given distance d' apart in medium 1 bears to the force between A and C at the same distance d' apart in medium 2.

If F_{ac} denotes the force between the poles A and C in any medium at any distance d apart, and F_{bc} the force between the poles B and C in the same medium at the same distance d apart, then, whatever this distance d, and whatever the medium, F_{ac} bears to F_{bc} a certain definite and constant ratio, and this ratio is unaltered if the pole C is exchanged for any other similar (concentrated) pole.

Hence there is associated with each of the two poles A and B a constant, which we shall call its *pole strength*, or *quantity of magnetism*, such that the force between one of these poles and a third concentrated pole is proportional to this constant or pole strength. But the same thing is true of the third pole also. Hence the force between two poles is proportional to the strength of each, that is to the product of their pole strengths. The strength of a pole will de denoted by m with a subscript to identify the pole when necessary.

Putting the above results together, we have, if F denotes the force between two concentrated poles of strengths m_1 and m_2 when the distance between them is L, the *law of Coulomb*:

$$F = Am_1m_2/\mu L^2 \tag{1}$$

μ being a constant of the medium in which the poles are immersed, called its *inductivity* (analogous to electric permittivity), and A a constant depending on the units in which all the other quantities are expressed.

If, as in this book, the c.g.s. system of mechanical units is adopted, if μ is expressed in terms of the inductivity of free æther (denoted by μ_0) as *unit inductivity*, and if A is put equal to $+1/4\pi$, m_1 and m_2, are, by definition, expressed in terms of the

rational electromagnetic (*REM*) *unit magnetic pole strength.* With these conventions, the above equation (1) becomes

$$F = m_1 m_2 / 4\pi\mu L^2 \qquad (2)$$

If m_1 and m_2 are poles of the same kind (both positive or both negative), F is positive or repulsive; if the signs of the poles are opposite, F is negative, or the force is one of attraction.

From the nature of the conditions assumed above it is clear that the law can not be established rigorously by direct experiment. The best proof of the law is the general agreement between experiment and a magnetic theory based largely upon the law. The physical reason for the existence of the law of inverse squares is similar to that for the law of inverse squares in electrostatics, as will be apparent when the magnetic flux and Gauss's theorem for magnetism have been discussed (§§ 13 and 14).

7. The Magnetic Field. Magnetic Intensity. Any region in which a magnetic pole is acted upon by a mechanical force in virtue of its magnetism is called a *magnetic field.* Such a field is the neighborhood of a magnetic pole, or the region about the earth, or the region about a wire carrying an electric current.

It follows from experiment that the force F acting upon a concentrated magnetic pole at any point of a magnetic field is proportional to its pole strength m (provided that the distribution of magnetism originally accompanying the field remains sensibly unaltered on the introduction of the pole). That is,

$$F = Hm \qquad (3)$$

The proportionality factor H is called the *magnetic intensity* at the point. H is evidently a vector specifying the state of the field at the point, its direction being that of the force upon a positive pole (or opposite to that of the force upon a negative pole), and its magnitude the magnitude of the force upon unit pole placed at the point.

When F is expressed in dynes and m in the *REM* unit, $H = F/m$ is, by definition, expressed in the *REM unit magnetic intensity.*

(2) is a particular case of (3).

The term *magnetic field* is also used to denote the collective intensity in a region instead of the region itself. The *direction of the field* at any point is then the direction of the intensity, and the *strength of the field* is the magnitude of the intensity.

8. The Strength of a Distributed Pole. By (3), which may be written

$$m = F/H \tag{4}$$

the strength of *any* magnetic pole when placed in a uniform field may be defined as the ratio of the force acting upon the pole to the magnetic intensity of the field (which can be determined by (3) with a concentrated pole). A general definition of pole strength consistent with (2) and (4) is given in § 21.

9. The Positive and Negative Poles of a Magnet Have the Same Numerical Strength, or the total quantity of magnetism in any magnet is zero. For if a magnet is placed on a float in a vessel of water, so as to be perfectly free to move in any direction, it experiences no translatory force in any direction when the surrounding field is that of the earth alone. Thus the force upon the positive pole is exactly equal and opposite to the force upon the negative pole. Hence, since the intensity of the earth's field throughout the region occupied by the magnet (and, in general, throughout a much larger region) is sensibly uniform, the pole strengths are equal and opposite by (4), and their algebraic sum is zero. See also § 21.

10. Magnetic Induction. Analogous to electric induction or displacement is a quantity called the *magnetic induction*, defined as the product of the inductivity and the magnetic intensity. Thus, if the magnetic induction is denoted by B, we have

$$B = \mu H \tag{5}$$

B is obviously a vector with the same direction as that of H, since μ (in isotropic media, which alone will be considered here) has no relation to direction.

When μ and H are expressed in *REM* units, $B = \mu H$ is said to be expressed in the *REM unit magnetic induction.*

A substance in which magnetic induction exists, that is, a substance which supports a magnetic field, is said to be *magnetised* or to be in a state of *magnetisation.* If the induction and inductivity are uniform throughout, the magnetisation is said to be *uniform.* The intensity of magnetisation is defined in § 21.

11. Lines and Tubes of Magnetic Intensity and Induction are defined in exactly the same way as lines and tubes of electric intensity and induction, except that magnetic quantities are substituted for electric throughout.

12. The Superposition of Magnetic Fields. The statements of § 12, I., and those of the paragraph following (2), § 11, I., with reference to the electric field hold also for the magnetic field, except that the medium supporting a magnetic field never breaks down as a result of magnetic stress (another illustration of the non-existence of magnetic conductivity).

13. The Magnetic Flux, Φ, across a surface S is defined in exactly the same manner, analogous terms being substituted, as the electric flux, § 22, I., that is, as the integral over the surface of the normal component of the induction. Thus

$$\Phi = \int B \cos\theta \, dS = \int \mu H \cos\theta \, dS \quad (6)$$

θ denoting the angle between B or H and the normal to dS.

14. Gauss's Theorem. The magnetic flux Φ outward across a closed surface S surrounding any number of *concentrated* magnetic poles of total strength m is equal to m.* This follows for an infinite or finite region containing a homogeneous isotropic medium (μ = constant, except within the magnets, whose vol-

* With the qualification made below. In every case, however, the *flux of magnectic intensity* outward across a closed surface surrounding a real pole distributed in any manner, multiplied by the inductivity of the medium surrounding the magnet, is equal to the strength of the pole. For the analogous electric case see VI. and IV.

umes are supposed negligible) from (2), (3), (5), and (6), and a process of reasoning exactly analogous to that employed in establishing the corresponding theorem in electrostatics.

The theorem holds only for concentrated poles, the flux from a distributed pole being, in general, very different from m. (See § 8, VI., for the corresponding electric case.)

The strength of a tube of induction. It follows from Gauss's theorem exactly as in electrostatics that the flux across every diaphragm of a given tube of magnetic induction in a homogeneous isotropic medium is the same. The magnitude of this flux is called the *strength* of the tube. A *unit tube* is a tube whose strength is unity.

We here assume, as in strict accord with experiment, that the strength of a tube is constant throughout its length whatever media it may traverse, whether the field is a pure magnetic field or an electromagnetic field (XII.). For a pure magnetic field (in which there is no intrinsic magnetisation) this result follows from considerations exactly similar to those adduced to establish the analogous proposition in electrostatics, except that two concentrated permanent magnetic poles, one in each medium, must be employed instead of the closed electric condenser.

In deriving Gauss's theorem the (infinitely small) volumes of the magnets and all their contents were neglected. It must always be remembered, however, that, as stated in § 2, magnetic poles are the analogues of the poles of electrets, not of true electric charges. We may, for convenience, consider only the flux from (or to) a pole in the surrounding medium, as we have just done; but we must remember that the same quantity of flux which emanates from a magnet at its positive pole enters the magnet again at its negative pole, making the total flux across any closed surface surrounding a single pole (or any number of poles) zero.

That this statement is correct for a magnet with concentrated poles follows from Gauss's theorem and the fact that if any magnet is broken across its axis into any number of pieces, each

piece is a magnet with its positive and negative poles equal in strength (numerically) and pointing in the same directions as the corresponding poles of the original magnet.

When both the original magnet and these pieces are very long and thin, as they must be to have approximately concentrated poles, the pole strengths of the pieces and the original magnet are sensibly equal.

We here *assume* that all tubes of magnetic induction are closed like the tubes just considered, whether in a pure magnetic field or in an electromagnetic field. This assumption is in strict accord with experiment and with a more general definition of magnetic induction given in Chapter XIII. Thus there is nothing in magnetism analogous to the discontinuity of electric displacement, or true electric charge.

Applying the above results to the element of volume at a point, we get, obviously,

$$\operatorname{div} B = 0 = \operatorname{div} \mu H \tag{7}$$

the flux into any element of volume across a part of its surface being equal to the flux out of the volume across the rest of the surface.

15. Magnetomotive Force or Gaussage. Magnetic Potential and Equipotential Surfaces. The line integral of magnetic intensity, $\int H \cos \theta \, dL$, along a path L from a point P_1 of a magnetic field to a point P_2 is called the *magnetomotive force* (m.m.f.) or *gaussage* from P_1 to P_2 along the path L. When this integral is the same along every path from P_1 to P_2 it is also called the *difference of magnetic potential* between P_1 and P_2, or the *fall of magnetic potential* from P_1 to P_2. From a process of reasoning similar to that of § 17, I., this is evidently the case in the field of a magnet (unaccompanied by electric currents).

The *unit gaussage* is the gaussage which exists between two points when unit work must be done to transfer a unit magnetic pole from one to the other.

The fall of magnetic potential from a given point P to a point at an infinite distance from all magnetic poles is called the *magnetic potential* at P, and will be denoted by Ω. This symbol will also be used to denote m.m.f. in the more general case.

A surface which is everywhere normal to the magnetic intensity is called a magnetic *equipotential surface*.

16. The Mapping of Magnetic Fields. A magnetic field can be completely mapped out by a system of tubes of induction of equal strength or by a system of equipotential surfaces between the successive surfaces of which the gaussage is constant. (See §§ 20 and 25, I.)

Maxwell's method of drawing such systems of tubes and surfaces applies equally to the electric and magnetic cases. (See §§ 7, 11, 13 and 14, II.)

17. Magnetic Conductors. An *imaginary* substance, analogous otherwise to an electric conductor, within whose volume there is no magnetic induction or intensity when immersed in a static magnetic field, and at whose surface all lines of magnetic intensity and induction are therefore discontinuous normally, is called a *magnetic conductor*. It will appear later (XVI.) that a perfect electric conductor behaves like a substance of *zero inductivity* and can under no circumstances support an electric or magnetic field.

The (imaginary) true magnetic charge and surface density upon such an imaginary surface are defined as the magnetic flux from surface and the induction at the surface, respectively, in a manner exactly analogous to that followed in electrostatics.

18. Magnetic Energy Density, Magnetic Tension, and Magnetic Pressure. From the strict mathematical analogy existing between the strength of a concentrated permanent magnetic pole and a concentrated electric charge or the concentrated pole of a permanent electret, magnetic inductivity and electric permittivity, magnetic intensity and electric intensity, magnetic induction and

electric induction or displacement, it follows that, when μ is constant,

(1) The magnetic energy per unit volume at any point of a magnetic field is

$$T = \tfrac{1}{2} BH = \tfrac{1}{2} \mu H^2 = \tfrac{1}{2} B^2/\mu \tag{8}$$

(2) There is a tension parallel to the intensity equal to

$$T = \tfrac{1}{2} BH = \tfrac{1}{2} \mu H^2 = \tfrac{1}{2} B^2/\mu \tag{9}$$

(3) There is a pressure in every direction normal to the intensity equal to

$$T = \tfrac{1}{2} BH = \tfrac{1}{2} \mu H^2 = \text{etc.} \tag{10}$$

It follows also, whether μ is constant or not, that the work per unit volume done in magnetising a substance from a value of the induction $B = B_1$ to a value $B = B_2$ is

$$dW/d\tau = \int_{B_1}^{B_2} HdB \tag{11}$$

which reduces to (8) when μ is independent of H. This expression does not give the change in the magnetic energy when hysteresis (§ 39, XIII.) is present, by far the greater part of the energy used during the magnetising process being in most cases dissipated.

(8) and (11) will be demonstrated later (§§ 12 and 29, XIII.) in a different manner.

19. Magnetic Energy. Permeance. Reluctance. In exactly the same manner, or by direct integration from (8), the energy contained in the volume T of a tube of magnetic induction of strength Φ between two equipotential surfaces between which the m.m.f. is Ω is

$$W = \tfrac{1}{2}\Phi\Omega \tag{13}$$

if μ is constant (independent of H); and

$$\begin{aligned} W = \int T d\tau = \int\int HdBd\tau = \int\int\int HdLdSdB \\ = \int\int HdL \cdot SdB = \int \Omega d\Phi \end{aligned} \tag{14}$$

in the general case, assuming no dissipation of energy. (14) gives the work done in magnetisation in every case.

The ratio of the magnetic flux Φ through the tube to the m.m.f. Ω between the two equipotentials is called the *permeance*, and its reciprocal the *reluctance*, of this portion of the tube. Thus, if the permeance is denoted by P and the reluctance by R,

$$P = 1/R = \Phi/\Omega \tag{15}$$

The combination of (13) and (15) gives for the energy W,

$$W = \tfrac{1}{2}\Omega\Phi = \tfrac{1}{2}P\Omega^2 = \tfrac{1}{2}\Phi^2/P = \tfrac{1}{2}\Omega^2/R = \tfrac{1}{2}R\Phi^2 \tag{16}$$

The electric analogue of permeance is evidently *permittance*.

[In the irrational systems of units also, Chapter XIV., P and R are defined by (15), the electrical analogy, requiring the introduction of 4π, not being strictly maintained.]

20. The Laws of Refraction of Lines of Intensity and Induction. It follows also, by a procedure exactly analogous to that of IV., § 2, or by inspection of the final equations of IV., § 2, and the analogies mentioned in § 2, that in crossing an interface from a medium 1 to a medium 2 a line of magnetic intensity or induction is refracted in such a way that

I. The incident and refracted lines are in the same plane perpendicular to the interface at the point of incidence; and that

II. The ratio of the tangent of the angle of incidence to the tangent of the angle of refraction is a constant for the given media (when the inductivity ratio is constant) and equal to the ratio of the two inductivities.

The equivalent equations are

$$H_1 \sin\theta_1 = H_2 \sin\theta_2 \text{ (tangential intensity continuous)} \tag{17}$$

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \text{ (normal induction continuous)} \tag{18}$$

$$\tan\theta_1/\tan\theta_2 = \mu_1/\mu_2 \tag{19}$$

21. Magnetic Surface and Volume Density, Intensity of Magnetisation, etc. Proceeding in the same manner and following IV., §§ 3–4, we get what follows.

The normal discontinuity in the magnetic intensity at the interface separating two media 1 and 2 (H_1 and H_2 being reckoned positive when directed from medium 2 to medium 1) is

$$H_1 \cos \theta_1 - H_2 \cos \theta_2$$

which is equal to $[(\mu_2 - \mu_1)/\mu_1\mu_2]B_2 \cos \theta_2$ when there is no intrinsic magnetisation (§ 22).

The *magnetic surface density* (analogous to *apparent* electric surface density) at the interface is

$$\sigma' = \mu_1(H_1 \cos \theta_1 - H_2 \cos \theta_2) = (B_2 - \mu_1 H_2) \cos \theta_2 \quad (20)$$

which is equal to $[(\mu_2 - \mu_1)/\mu_2]B_2 \cos \theta_2$ when there is no intrinsic magnetisation present. [In the irrational systems of units, Chapter XIV., σ' is defined by the relation

$$4\pi\sigma' = \mu_1(H_1 \cos \theta_1 - H_2 \cos \theta_2)]$$

The *intensity of magnetisation* of medium 2 with respect to medium 1 is, by definition,

$$J = B_2 - \mu_1 H_2 \quad (21)$$

which is equal to $[(\mu_2 - \mu_1)/\mu_2]B_2$ when there is no intrinsic magnetisation. An equivalent definition of J is the *magnetic moment of medium 2 per unit volume* (§ 23, and § 12, IV.). (21) shows that J is the difference between the actual induction in 2 and the induction which would exist there if μ_2 were equal to μ_1 with the same value of the intensity. [In the irrational systems of units (Chapter XIV.), J is defined by the equation $4\pi J = B_2 - \mu_1 H_2$.]

If we write $B_2 = \mu_2 H_2$, (21) may be written

$$J = (\mu_2 - \mu_1)H_2 = \kappa H_2 \quad (22)$$

κ, which is written for $(\mu_2 - \mu_1)$, is called the *magnetic susceptibility* of medium 2 with respect to medium 1. [In the irrational system of units $\kappa = (\mu_2 - \mu_1)/4\pi$.]

(22) may be written

$$B_2 = \mu_2 H_2 = J + \mu_1 H_2 \quad (23)$$

(20) may be written

$$\sigma' = J \cos \theta_2 \tag{24}$$

The *quantity of magnetism* upon a surface S, or the *pole strength* of the *surface* S, is

$$\int \sigma' dS = \int J \cos \theta_2 dS \tag{25}$$

which may be written equal to $[(\mu_2 - \mu_1)/\mu_2]\Phi$ when there is no intrinsic intensity.

The *magnetic volume density* in a region where μ varies from point to point is

$$\rho' = \mu_1 \operatorname{div} H \tag{26}$$

(In the irrational systems of units, Chapter XIV., $4\pi\rho' = \mu_1 \operatorname{div} H$.)

The total quantity of magnetism in a volume τ is

$$\int \rho' d\tau = \mu_1 \int \operatorname{div} H \, d\tau \tag{27}$$

The total quantity of magnetism within a volume τ and upon its surface S is

$$m = \int \sigma' dS + \int \rho' d\tau = \int J \cos \theta_2 \, dS + \mu_1 \int \operatorname{div} H \, d\tau \tag{28}$$

The force upon a magnetic pole of strength m in a field of intensity H is

$$F = Hm \tag{29}$$

The intensity at a point P distant L from the element dm of a pole of strength dm "due" to dm is directed along L and is equal to

$$dH = dm/4\pi\mu_1 L^2 \tag{30}$$

The magnetic potential at a point due to any magnetic distribution whatever is

$$\Omega = 1/4\pi\mu_1 \cdot \int dm/L \tag{31}$$

If medium 2 is uniformly magnetised, the magnetic flux through the positive pole will be

$$\begin{aligned} \Phi &= \int B_2 \cos \theta_2 dS = \int J \cos \theta_2 dS + \mu_1 \int H_2 \cos \theta_2 dS \\ &= m + \mu_1 \int H_2 \cos \theta_2 dS \end{aligned} \tag{32}$$

which is equal to m only when the pole is concentrated. In isolated magnets H_2 is negative, hence Φ is less than m (except in the ideal case mentioned).

The complete developments § 7, IV., and § 1, V., are valid for the magnetic case, magnetic quantities being substituted for the analogous electric quantities.

22. Intrinsic Magnetisation, Etc. Corresponding to intrinsic electrisation, intrinsic electric intensity, etc., are *intrinsic magnetisation,* of which we have an example in every magnet, *intrinsic magnetic intensity* or *force* (denoted by h) maintaining the induction, etc. Intrinsic magnetic phenomena do not appear in most substances, but are far more marked than the corresponding electric phenomena in others, notably iron, nickel, and cobalt, reaching their maximum development in hard steel. The magnetisation of these substances will be briefly discussed in §§ 35–39, XIII.

Permanent Magnets. A permanent magnet is a magnet whose pole strengths at any definite temperature are constant independently of the time, of the nature of the surrounding medium, and of the proximity of other magnets and electric currents. Such a magnet is purely ideal, but approximately permanent magnets can be made of long cylinders of properly tempered hard steel. Such a magnet, if not brought into too strong magnetic fields, and if kept when not in use in an iron case will retain its moment constant in air, even when the ratio of its length to the square root of its cross-section is less than ten to one, for at least a year within one tenth per cent. (Klemencic, *Ann. der Physik,* XII., 174, 1901). From the discussion of the permanent electret, § 8, VI., it is clear that a magnet will be more nearly permanent the greater the ratio of its length to its diameter (or other linear dimensions perpendicular to its length). It is also evident that, other things being equal, the greater the coercive force (§ 25, XIII.) of a substance or the "harder" it is magnetically, the nearer will be its approach to permanency of magnetisation, when made up into magnets.

23. Pure Magnetic Fields. It is now evident that the magnetic fields for a great variety of ideal distributions can be obtained from the analogous electric fields already discussed in previous chapters, by substituting magnetic conductors for electric conductors, inductivity for permittivity, true magnetic charge for true electric charge, magnetic pole strength for fictitious electric charge, magnetic intensity for electric intensity, magnetic induction for electric displacement, intensity of magnetisation for intensity of electrisation, etc. Since the concentrated poles of permanent electrets can be substituted for concentrated electric charges, the electrisation within the volume of the electrets being neglected (§ 8, VI.), concentrated electric charges can be replaced in the magnetic analogues by concentrated magnetic poles, the magnetic induction within the volumes of the linear magnets being neglected. The approximate effects of magnetic conductors, except as regards the continuity or discontinuity of the magnetic induction, can be obtained by using substances of great inductivity. (The value of μ for soft iron may, with vibration, reach nearly 80,000 μ_0. See Ewing, *Phil. Trans.*, 1885.)

Thus, Fig. 14 represents the field of a single concentrated magnetic pole, the induction within the magnet, supposed infinitely thin, being neglected, and the opposite pole being supposed infinitely remote.

Fig. 22 shows the field surrounding a single magnet with concentrated poles.

Fig. 60 shows the field about a concentrated magnetic pole in an infinite medium of inductivity μ_1 separated by a plane interface from an infinite medium of inductivity μ_2 for the particular case in which $\mu_2/\mu_1 = 4$, the field within the magnet being neglected and its opposite pole being infinitely remote.

Figs. 60, 61, and 62 show the magnetic field in and around a sphere of inductivity μ_2 placed in an infinite medium of inductivity μ_1 supporting an originally uniform field for the three cases in which $\mu_2/\mu_1 = 0$, 3, and infinity.

Fig. 63 shows the field of an infinite circular cylindrical shell of inductivity μ_2 placed in an infinite medium of inductivity μ_1 supporting (originally) a uniform magnetic field perpendicular to the axis of the cylinders for the particular case in which $\mu_2/\mu_1 = 10$. §§ 14–15 were in fact developed largely on account of the magnetic fields exactly analogous to the electric fields there investigated. The table (Table I., §§ 14–15) is constructed with values of c_2/c_1 or μ_2/μ_1 common in magnetism but never occurring in electrostatics. The use of spherical and cylindrical iron shells as magnetic screens in protecting galvanometer magnets from external fields is mentioned in § 30, XII.

Fig. 67 shows the field of a uniformly magnetised isolated sphere, Fig. 28, with the additions and modifications indicated in § 6, VI., that of an infinite circular cylinder uniformly magnetised at right angles to its length, etc., etc., every electric field having its magnetic analogue, either ideal or real.

24. Resultant Magnetic Poles, Magnetic Axis. The total forcive upon a magnet in a uniform field, as the earth's magnetic field, is a torque tending to bring its axis into the direction of the field. It is evident that the axis of the magnet, as defined in § 4, is the direction of the straight line drawn from the center of the parallel forces due to the uniform field, as the earth's field, on all the elements of the distributed negative pole to the center of the opposite parallel forces upon all the elements of the positive pole.

These centers may be regarded as the positions of the *resultant* poles of the magnet; but it must be remembered that if the forces upon all the elements are not *parallel*, as they are when the magnet is in a uniform field, the positions of the centers, or points of application of the resultant forces, are different, and different for every different field. If the magnetism were uniformly distributed through spheres or in concentric spherical shells, or over spherical surfaces, at the two ends of the magnet, then it could easily be shown, as in the corresponding cases in

electrostatics and gravitation, that the field outside the spheres, and the resultant forces, would in all cases be the same as if the poles were concentrated at the centers of the spheres. The forces between the poles of two magnets at a distance, however, will be approximately the same as if the poles were concentrated in the positions of the resultant poles for a uniform field, just as the gravitational force between two masses of irregular shape, when the distance between them is considerable, so that the field due to each is nearly uniform at the other, is nearly the same as if the masses were concentrated at their centers of gravity or centers of mass.

For convenience, the straight line drawn from the negative resultant pole to the positive resultant pole, as well as its direction, will be called the *axis* of the magnet. If the magnet is a cylinder of symmetrical cross-section and magnetised symmetrically with respect to its geometrical axis, the geometrical and magnetic axes will coincide.

25. The Torque Upon a Magnet in a Uniform Field. Magnetic Moment. Let m denote the magnitude of the strength of each pole, L the distance between the resultant poles, H the intensity of the uniform field, and θ the angle between the direction of the axis and the direction of the uniform field. The torque tending to diminish θ is evidently $2mH\frac{1}{2}L \sin \theta$, and the torque tending to increase θ, or the torque measured in the same direction as that in which θ is measured, is

$$T = -2mH\tfrac{1}{2}L \sin \theta = -mLH \sin \theta = -MH \sin \theta \quad (33)$$

if we put $mL = M$.

The quantity $M = mL = T/H \sin \theta$ is called the *magnetic moment* of the magnet (analogous to the electric moment of an electret).

26. Gauss's Method of Determining Simultaneously the Moment M of a Magnet and the Horizontal Component H of the Earth's Magnetic Intensity. I. Determination of $M\mathsf{H}$. The magnet, A, of moment M is first suspended with its axis horizontal by a ver-

tical fiber as free from torsion as possible, and set to vibrating through a very small horizontal arc. The time T of a complete vibration of the magnet, and its moment of inertia K about the axis of vibration, are then determined. Then the product $M\mathbf{H}$ is given by the equation

$$M\mathbf{H} = 4\pi^2 K/T^2 \tag{34}$$

to a high degree of approximation.

For, since $T = Kd^2\theta/dt^2$, (33) gives for the equation of motion of the magnet

$$Kd^2\theta/dt^2 + M\mathbf{H}\sin\theta = 0 \tag{a}$$

For very small amplitudes of vibration $\sin\theta = \theta$ very approximately and (a) becomes

$$Kd^2\theta/dt^2 + M\mathbf{H}\,\theta = 0 \tag{b}$$

which shows that the motion is simple harmonic in the period $T = 2\pi(K/M\mathbf{H})^{\frac{1}{2}}$, a relation identical with (34).

If the torsion of the suspension and the arc of vibration are large enough to have appreciable effects, they can be determined and allowed for by methods given in Maxwell's *Treatise*, §§ 452 and 738, and in laboratory manuals.

II. Determination of $M/\mathbf{H}$. The magnet A is removed from the suspension and mounted with its axis horizontal and perpendicular to the magnetic meridian in a line passing through the center of a very small magnet B suspended by a fiber of negligible torsion (the torsion, if not negligible, can be determined and allowed for), the whole forming a *magnetometer*, in the position occupied by the center of A in I., and the distance R_1 between the centers of A and B is measured. R_1 must be great in comparison with L, the distance between the resultant poles of A, which is approximately equal to the length of the magnet.

The magnetic intensity, H_1, at B due to the poles of A is perpendicular to $\mathbf{H}$. Hence the resultant horizontal intensity at B makes with the magnetic meridian the angle θ_1 whose tangent is

$$\tan\theta_1 = H_1/\mathbf{H} \tag{35}$$

The axis of the magnet B, in the meridian before the introduction of A, takes up a position parallel to the resultant intensity. That is, on the introduction of A, B is deflected through the angle θ_1, which must be measured.

Denoting the numerical strength of each pole of A by m and assuming the magnetisation of A to be symmetrical, so that the distance of each resultant pole is $L/2$ from the center of A, we have for H_1

$$H_1 = m/4\pi\mu \cdot [1/(R_1 - \tfrac{1}{2}L)^2 - 1/(R_1 + \tfrac{1}{2}L)^2)]$$
$$= M/2\pi\mu R_1^3 \cdot (1 - L^2/4R_1^2)^{-2}$$
$$= M/2\pi\mu R_1^3 \cdot (1 + L^2/2R_1^2 + 3L^4/16R_1^4 + \cdots)$$
$$= M/2\pi\mu R_1^3 \cdot (1 + L^2/2R_1^2) = \mathbf{H} \tan \theta_1 \qquad (c)$$

if the fourth and higher powers of L/R_1 are neglected.

Since (c) contains the unknown quantity L in addition to M and $\mathbf{H}$, the experiment is repeated with a different distance, R_2, between the centers of A and B, and the corresponding angle of deflection θ_2 is determined. Then

$$M/2\pi\mu R_2^3 \cdot (1 + L^2/2R_2^2) = \mathbf{H} \tan \theta_2 \qquad (d)$$

Eliminating L from (c) and (d), we obtain

$$M/\mathbf{H} = 2\pi\mu(R_1^5 \tan \theta_1 - R_2^5 \tan \theta_2)/(R_1^2 - R_2^2) \qquad (36)$$

From (34) and (36) both M and $\mathbf{H}$ can be calculated, since μ for air is known and sensibly equal to $\mu_0 = 1$.

To eliminate the error arising from the (possible) lack of symmetry in the magnetisation of A, and errors in determining θ_1, θ_2, R_1, and R_2, the angles are read for each value of R, with A either east or west of B, first with the one pole of A toward B, then with the other pole toward B. Then A is placed on the opposite side of B and the angles read, for both directions of the axis of A, with the same values of R_1 and R_2. The mean values of θ_1 and θ_2 derived from all the readings are used in the calculation of $M/\mathbf{H}$ by (36).

27. The Comparison of Two Intensities. By vibrating the same magnet successively in two fields of intensities H_1 and H_2, we obtain from (34),

$$H_1/H_2 = T_2^2/T_1^2 \tag{37}$$

By using the same magnet A to deflect a small magnet B in two different fields of intensities H_1 and H_2, A being placed with reference to the center of B and the direction of the field in each case as described in II., and the distance between the centers of the magnets being the same in both cases, we obtain from (35)

$$\tan\theta_1/\tan\theta_2 = H_2/H_1 \tag{38}$$

The Comparison of Two Magnetic Moments. By vibrating in the same field two magnets of moments M_1 and M_2, and moments of inertia about the axis of suspension K_1 and K_2, we obtain from (34)

$$M_1/M_2 = K_1T_2^2/K_2T_1^2 \tag{39}$$

The comparison may also be made by means of equation (36).

CHAPTER XII.

THE MAGNETIC FIELD OF THE CONDUCTION CURRENT.

1. Relation Between the Direction of a Current and the Direction of its Lines of Intensity. An electric conduction current is invariably accompanied by a magnetic field surrounding and penetrating into the conductor. In the case of a long straight circular cylindrical wire carrying a current and immersed in a medium of uniform inductivity the lines of magnetic intensity, as will be shown in § 15, are circles centered on the axis of the wire in planes perpendicular to the axis, and the direction of each line is related to the direction of the current as the rotation to the translation of a right-handed screw or as the direction of

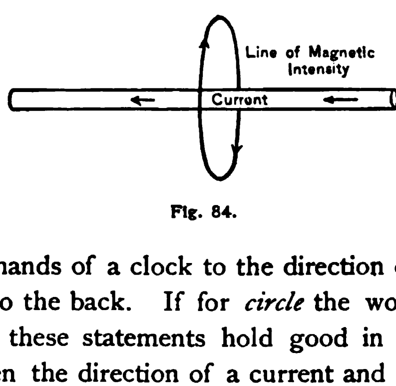


Fig. 84.

motion of the hands of a clock to the direction of a line drawn from the face to the back. If for *circle* the words *closed curve* are substituted, these statements hold good in all cases. The relation between the direction of a current and that of its lines of intensity is shown in Fig. 84.

2. Positive Directions Around and Through a Circuit. The *positive direction around a circuit* bears to the *positive direction through the circuit*, by definition, the same relation as the direction of rotation of a right-handed screw bears to its direction of

translation. Thus if AB, Fig. 85, is chosen as the positive direction through the circuit $CDEC$, the positive direction around the circuit is $CDEC$ as indicated by the arrows. If the direction

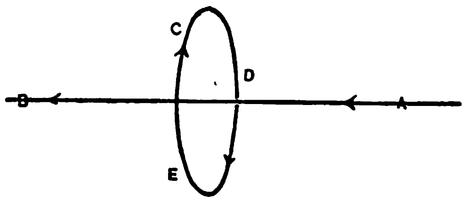


Fig. 85.

of the current in a circuit is chosen as the positive direction around the circuit, as will be done in this chapter, the magnetic flux connected with the current will always thread the circuit in the positive direction.

3. Vector Product of Two Vectors. Let A and B denote two vectors intersecting at an angle θ less than π. If C, a third vector, is equal in magnitude to the product $AB \sin\theta$, and is perpendicular to their plane and so directed that a right-handed

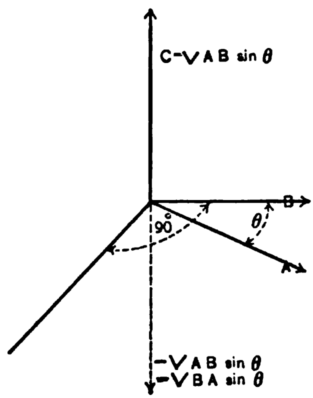


Fig. 86.

screw progressing along C in the positive direction would rotate from A to B, Fig. 86, then C is called the *vector product* of A and B, and is written

$$C = \mathbf{V}AB \sin\theta \qquad (1)$$

If C has the opposite direction, as indicated by the dotted line in the figure, it may, consistently with (1), be written

$$C = -\mathbf{V}AB \sin\theta = \mathbf{V}BA \sin\theta$$

4. The Law of Ampère. If a very long and thin cylindrical conductor carrying a current is immersed in a uniform magnetic field perpendicular to its direction, the two fields will combine to form the resultant field shown in cross-section in Fig. 87, the resultant field being stronger where the component intensities have the same general direction, and weaker where their general directions are opposite. The pressures and tensions in the field will then give rise to a force upon the conductor directed from the stronger to the weaker part of the resultant field, and thus, as the figure shows, perpendicular to the wire and to the originally uniform field. In the figure the conductor, perpendicular to the paper, is shown as a small circle; the current flows downward and the original field is directed to the left, or the current flows upward and the original field is directed to the right. In either case the force upon the conductor is directed toward the top of the page. This is a simple case coming under the *law of Ampère*, to which we proceed.

Consider a wire of negligible cross-section carrying a current I in a magnetic field of any kind. Let dL denote the element of length of the wire at any point P, and let B denote the magnetic induction at the point.* Let θ denote the angle between I and B. (For convenience we shall here treat I as if it, as well as the current density i, of which it is the surface integral, were a vector.) Then the results of experiment may be summed up in the following statement, which constitutes *Ampère's law* in the first form:

The wire is acted upon by a force F or a torque T which may be found by assuming each element of the wire of length dL to be acted upon by a force

$$dF = 1/a \cdot \mathbf{V}IB \sin\theta \cdot dL \qquad (2)$$

* See Lord Rayleigh, *Philosophical Magazine*, June, 1898.

where a is a constant depending on the units in which the other quantities in (2) are expressed, and taking the vector summation or integration of all the elementary forces dF along the part of

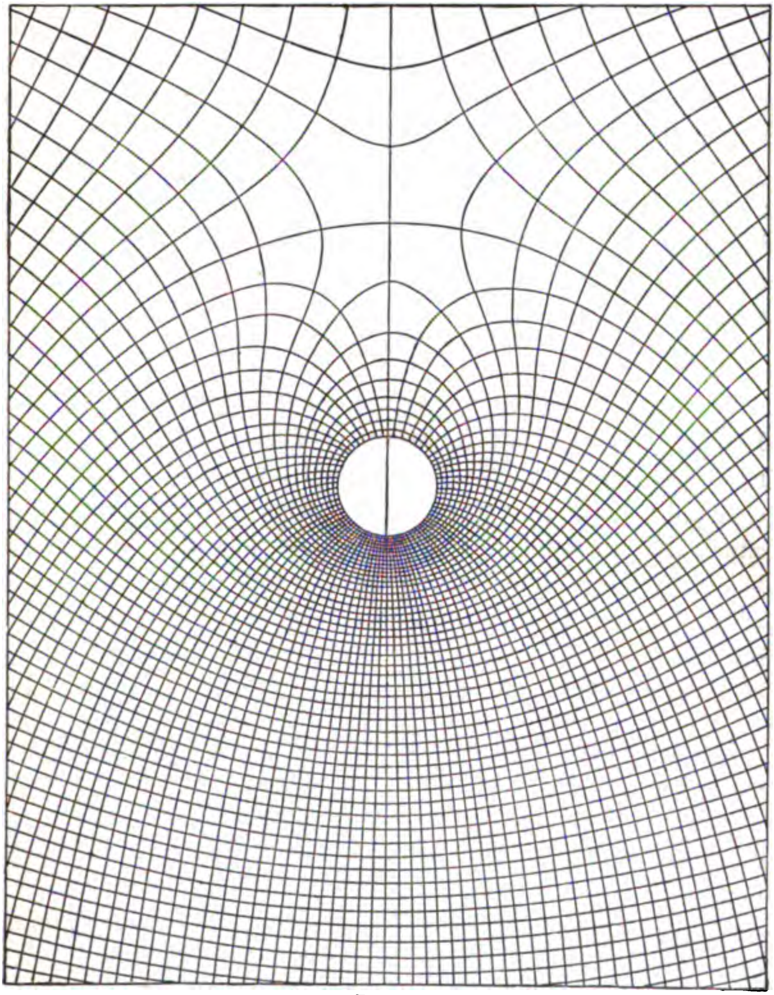


Fig. 87.

the wire considered, or of all the elementary torques dT arising from the forces dF. The force upon dL is always from the stronger to the weaker part of the resultant field.

If the wire is not so thin that its cross-section can be neglected, it may be considered split up into elementary current tubes and the summation or integration applied to all the elements of each tube.

5. The Rational Electromagnetic Unit Current. If in the above equation dF and dL are expressed in c.g.s. units, I in the *RES* unit, and B in the *REM* unit, a is found to have the magnitude 3×10^{10} (see § 4, XIV.). The *REM unit current* is defined as a unit current a times as great as the *RES* unit current. Hence, if we express I in terms of this unit, $1/a$ [if a is *assumed* to have zero dimensions (XIV.)] will disappear from the above equation, which thus becomes

$$dF = \mathbf{V} IB \sin \theta \cdot dL \tag{3}$$

3. Electric Units in the Rational Electromagnetic Unit System. If the electric current is expressed in terms of the electromagnetic unit, however, the relations $q = It$, $dH/dt = RI^2$, $\Psi = RI$, etc., will not remain true unless the units of charge, resistance, e.m.f., etc., are redefined. Units so chosen as to make these relations, or any of the relations which precede or follow, correct when I, or any other quantity occurring in the relations, is expressed in the *REM* unit current are defined as the *REM unit charge*, *resistance*, etc.

Magnetic and Electromagnetic Units in the Rational Electrostatic System. We have not hitherto defined any pure magnetic unit, such as magnetic pole strength or magnetic intensity, in the electrostatic system; but from the above definitions of the *REM* unit current, charge, etc., in terms of the *RES* units we can proceed immediately to such definition: Units so chosen as to make (3) and all the equations which follow in this work, as well as all those of Chapter XI., correct when I, or any other quantity occurring in the equations, is expressed in the *RES* unit are defined to be the *rational electrostatic units* of the quantities concerned.

Henceforth every equation will be expressed in one system of units throughout, like all preceding equations except (2), all the quantities being expressed in *RES* units, or else all in *REM* units.

The general subject of electric units is discussed in Chap. XIV.

7. The Force Upon a Straight Wire in a Uniform Magnetic Field. As a particular case falling under Ampère's law, we shall consider first a straight wire in a uniform magnetic field, and shall find the force upon a length L of the wire.

(1) If the wire is parallel to B, $\sin\theta = 0$ everywhere, hence $F = 0$.

(2) If the wire is perpendicular to the field, $\sin\theta = 1$ everywhere, and

$$F = IBL \tag{4}$$

The resultant field (a uniform field and the field of § 14 superposed) and the direction of F are shown in Fig. 87 (from Maxwell's *Treatise*, § 496).

(3) If the wire makes an angle θ with B

$$F = IB\sin\theta L = IBL\sin\theta \tag{5}$$

The force is the same, both in magnitude and direction, as would be exerted on the wire in a field $B\sin\theta$ perpendicular to L, or upon a wire of length $L\sin\theta$ in a field perpendicular thereto.

8. A Linear Circular Circuit in the Radial Field from a Concentrated Magnetic Pole of Strength m Placed (1) at its Center. (Fig. 88.) The magnetic induction and the electric current in the lin-

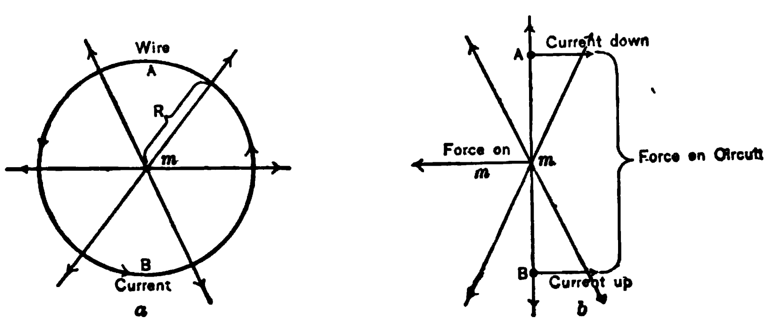


Fig. 88.

ear circuit are obviously everywhere perpendicular. The force upon the circuit is

$$F = I\int B \sin\theta \, dL = I(m/4\pi R^2)2\pi R = mI/2R \qquad (6)$$

The direction of the force is downward into the paper in Fig. 88*a* and to the right in Fig. 88*b*, when the current has the direction indicated and m is positive. If the sign of either is changed, the direction of F is reversed.

(2) On the Axis of the Circle at the Distance d from its Center. If the pole is placed on the axis of the circle at the distance d

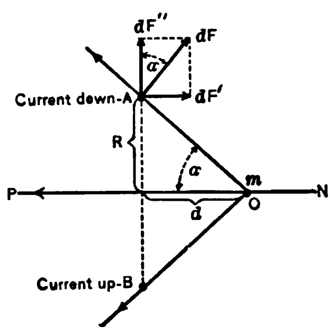


Fig. 89.

from its center, Fig. 89, $B \sin\theta = B = m/4\pi(d^2 + R^2)$, and the force upon each element of length dL of the wire is

$$dF = IB \sin\theta dL = Im/4\pi(d^2 + R^2)\cdot dL$$

This force may be resolved into two components, one

$$dF' = dF \sin\alpha = mIR/4\pi(d^2 + R^2)^{\frac{3}{2}}\cdot dL$$

in the direction of the axis; and the other

$$dF'' = dF \cos\alpha$$

in the direction of the radius. The second component tends to increase or diminish the radius of the wire according as the flux from the pole threads the circuit in the positive or negative direction through the circuit, but gives rise to no resultant force upon the circuit as a whole in any direction. The first component,

summed up for the whole length of the wire, gives, for the total force upon the wire in the direction of the axis,

$$F = \int dF' = mIR^2/2(d^2 + R^2)^{\frac{3}{2}}. \tag{7}$$

When $d = 0$, (7) reduces to (6).

9. An Infinite Straight Linear Wire in the Radial Field from a Concentrated Magnetic Pole of Strength m Distant d from the Wire. (Fig. 90.) In this case F has the magnitude

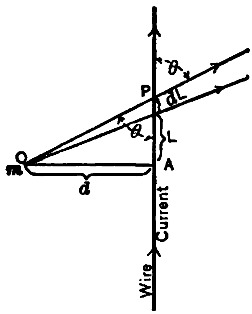


Fig. 90.

$$\begin{aligned} F &= I\int B \sin\theta\, dL = Im/4\pi \cdot \int 1/(L^2 + d^2) \cdot d/(L^2 + d^2)^{\frac{1}{2}} \cdot dL \\ &= Imd/2\pi \cdot \int 1/(L^2 + d^2)^{\frac{3}{2}} \cdot dL = Im/2\pi d \cdot \int_{\pi/2}^{0} \sin\theta\, d\theta \qquad (8) \\ &= Im/2\pi d \end{aligned}$$

since $L = d \cot\theta$.

The relative directions of F, B, and the current are obvious.

10. A Closed Plane Circuit of Any Form in a Uniform Magnetic Field. (Fig. 91.) Let the plane of the circuit be vertical and the field horizontal. Let z, x, be the coördinates of any point P of the circuit, referred to its highest point O as origin, the positive direction of Z being taken as the direction of the current at O, and the positive direction of X vertically downward. Let α denote the angle between B and the positive direction of the Z axis.

Let B be resolved into two components B sin α and B cos α perpendicular and parallel, respectively, to the plane of the circuit.

Owing to the component B sin α, there will be an outward force $dF_{xz} = IB \sin \alpha \, dL$ upon every element of the circuit per-

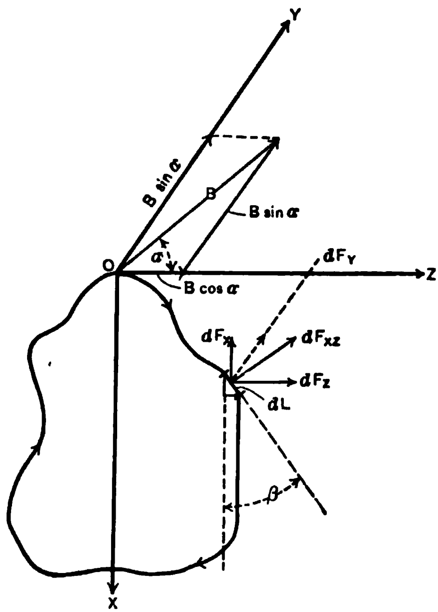


Fig. 91.

pendicular to its length dL and to B sin α in the plane of the circuit. Resolving this force along OX and OZ, we have, if dL (taken in the direction of the current) makes an angle β with OX,

$$dF_x = dF_{xz} \cos \beta = IB \sin \alpha \cos \beta \, dL = IB \sin \alpha \, dx$$

and

$$dF_z = dF_{xz} \cos (90^\circ + \beta) = - IB \sin \alpha \sin \beta \, dL = - IB \sin \alpha \, dz$$

By integration along the whole circuit

$$F_x = IB \sin \alpha \int dx = 0$$

and
$$F_z = - IB \sin \alpha \int dz = 0$$

Hence the forcive due to B sin α, if not zero, is a torque. Since

all the forces dF_{xz} are in the plane of the circuit, there can be no torque about an axis in this plane. Let therefore OY (the positive direction through the circuit), perpendicular to OX and OZ at O, be taken as the axis of moments. The force dF_{xz} on the element dL gives rise to a torque

$$dT = dF_x z - dF_z x = -IB \sin \alpha \, z dz - IB \sin \alpha \, x dx$$
$$= -IB \sin \alpha \, (z dz + x dx)$$

By integration around the circuit we obtain

$$T = \int dT = -IB \sin \alpha \left(\int z dz + \int x dx \right) = 0$$

Hence, so far as the component $B \sin \alpha$ is concerned, there is no resultant torque or force upon the whole circuit. Under the action of the force $dF_{xz} = IB \sin \alpha \, dL$, however, each element of the circuit tends to move outward or inward according as $\sin \alpha$ is positive or negative, *i. e.*, according as the flux of the uniform field threads the circuit in the positive or negative direction through the circuit; thus the circuit, if made of elastic material, would expand or contract.

The component $B \cos \alpha$ gives rise to a force dF_y on the element dL perpendicular to the plane XZ of the circuit. If OY is taken as the direction of a positive force,

$$dF_y = -IB \cos \alpha \cos \beta \, dL = -IB \cos \alpha \, dx$$

and the total force upon the circuit in the direction OY is

$$F_y = \int dF_y = -IB \cos \alpha \int dx = 0$$

Hence the forcive, if anything, is a torque. Since there is no force on any element in the direction OZ or ZO, there can be no torque about OY. The torque about OZ is

$$T_z = \int dT_z = \int x dF_y = -\int xIB \cos \alpha \, dx = -IB \cos \alpha \int x \, dx = 0$$

The torque about the axis OX is

$$T_x = \int dT_x = -\int z dF_y = IB \cos \alpha \int z dx$$
$$= IB \cos \alpha \int dS = IB \cos \alpha \, S$$

if S denotes the area of the circuit.

Thus the total forcive upon the circuit is a pure torque

$$T = IB \cos \alpha S \tag{9}$$

around a line in the plane of the circuit and perpendicular to the original magnetic field, the positive direction of the torque being always related to the positive direction of OX as the rotation to

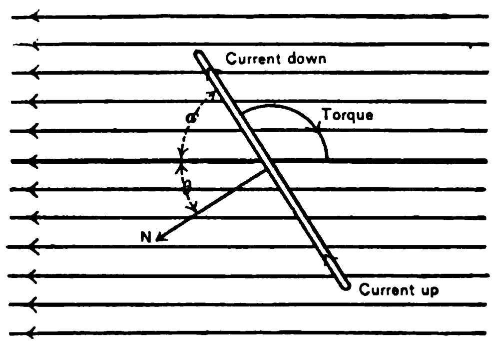


Fig. 92.

the translation of a right-handed screw with OX as axis. When the torque is positive it tends to increase the angle α (see Fig. 92).

If there are n turns in the circuit, the torque is n times as great.

If θ denotes the angle between the normal to the circuit (in the positive direction, § 2) and B, the torque tending to increase the angle θ is

$$T = -IBS \sin \theta = -I\mu SH \sin \theta \tag{10}$$

so that the circuit behaves like a magnet of moment $I\mu S$.

When $\theta = 0°$, $T = 0$, and the circuit is in stable equilibrium; and when $\theta = 180°$, $T = 0$, and the circuit is in unstable equilibrium. For in the former position the torque brought into existence by a slight change of θ will tend to restore equilibrium; while in the latter the torque developed by a similar change in θ will tend to increase the displacement.

When $\theta = 0$ and $T = 0$, the circuit encloses the maximum magnetic flux possible, a quantity BS due to the uniform field,

and that connected with its own current; the uniform flux, in this position of the circuit, being directed through the circuit in the positive direction.

When $\theta = 90°$ or $270°$, the circuit encloses no flux of the uniform field, and the torque is a maximum.

The circuit thus tends to move in such a way as to enclose the maximum flux possible.

We have already noted the tendency of a closed circuit carrying a current to expand or contract when placed in a radial field or a uniform field, according as the magnetic flux of this field threads the circuit in the positive or negative direction through the circuit. That is, the circuit expands or contracts according as the expansion or contraction will increase the flux through the circuit in the positive direction, or diminish the flux through the circuit in the negative direction. For the same reason (Ampère's law) the circuit would tend to expand if only in its own field, and thus to enclose as great a quantity of magnetic flux as possible.

Similar considerations would show that in every case a circuit carrying a current moves or tends to move in such a manner as to make the magnetic flux threading it as great as possible.

11. Magnetic Intensity Due to any Current Distribution.—From Ampère's law (first form) and the third law of motion we can find, for any point P, an expression for the magnetic intensity connected with any current distribution (Ampère's law, second form).

To do this imagine the conductor immersed in a radial field from a concentrated magnetic pole of strength m at the given point P distant r from dL, an element of length of the wire or current tube under consideration. Then the induction at dL due to the pole is $B = m/4\pi r^2$ directed radially from P. If we write $\underline{r}$ in the numerator to indicate the direction of B, $\underline{r}$ ($= r$ numerically) being measured from the pole to dL, this expression becomes

$$B = m\underline{r}/4\pi r^3$$

The force upon dL is therefore

$$dF = mI/4\pi r^3 \cdot \mathbf{V}\, dL\underline{r} \sin\theta$$

for a linear current, dL being measured in the direction of the current. Hence

$$F = mI/4\pi \times \text{vector integral of } 1/r^3 \cdot \mathbf{V} dL\underline{r} \sin\theta / r^3$$

Since F is the total force upon the circuit, there must be an equal and opposite force, $-F$, upon the pole at P. Hence, if H denotes the magnetic intensity at P due to the current,

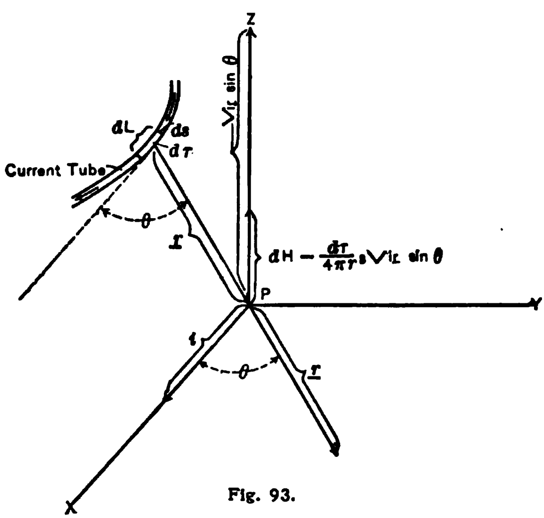


Fig. 93.

$$-F = Hm = mI/4\pi \times \text{vector integral of } (\mathbf{V}\underline{r}dL \ \sin\theta)/r^3$$

or, if r is measured from dL to the point P, instead of from P to dL,

$$Hm = mI/4\pi \times \text{vector integral of } (\mathbf{V}dL\underline{r} \sin\theta)/r^3$$

whence

$$H = I/4\pi \times \text{vector integral of } (\mathbf{V}dL\underline{r} \sin\theta)/r^3 \qquad (11)$$

The magnetic intensity at P is thus the same as if each element of the current (IdL) produced at P an intensity

$$dH = I/4\pi r^3 \cdot \mathbf{V}dL\underline{r} \sin\theta \qquad (12)$$

magnitude and direction ($\underline{r}$ being measured from dL to P); or, numerically, $I/4\pi r^2 \cdot \sin\theta\, dL$.

If the conductor has not a negligible cross-section, we have, clearly,

$$dH = I/4\pi r^3 \ (\mathrm{V} i \underline{r} \sin\theta) d\tau \tag{13}$$

and
$$H = I/4\pi r^3 \cdot \text{ vector integral } (\mathrm{V} i \underline{r} \sin\theta) d\tau \tag{14}$$

Equations (11)–(14) express Ampère's law in its second form.

It must be remembered that the *integral* force F and the *integral* intensity H, not the *elements* dF and dH, are all that experiment furnishes.

The relations between the quantities of equation (13) are shown in Fig. 93. dL, i, and r, are taken in the XY plane, and the direction of PX, perpendicular to the XY plane, is chosen to coincide with that of i and dL.

If i (or I) has everywhere one direction, then it is obvious from (12) or (13), that there is no component of magnetic intensity in this direction anywhere.

From all the above equations it is manifest that H does not depend in any way upon the inductivity of the medium in which the conductors are placed, provided that the medium is *homogeneous and isotropic* (so that the field from a concentrated pole anywhere is radial).

In what follows all media will be supposed homogeneous and isotropic unless the contrary is stated.

12. The Magnetic Field Around an Infinitely Long Straight Linear Conductor Carrying a Current I. From equation (12), or from the direct application of the third law of motion to (8), § 9, it is evident that the lines of intensity are circles centered on the wire and perpendicular thereto, the direction of the lines around the wire being related to the direction of the current as the rotation to the translation of a right-handed screw. The magnitude of the intensity is, by (11) or (8),

$$H = F/m = I/2\pi d \tag{15}$$

at a distance d from the wire.

Maxwell's plane diagram of a part of the field is shown in Fig. 16, Chapter II.

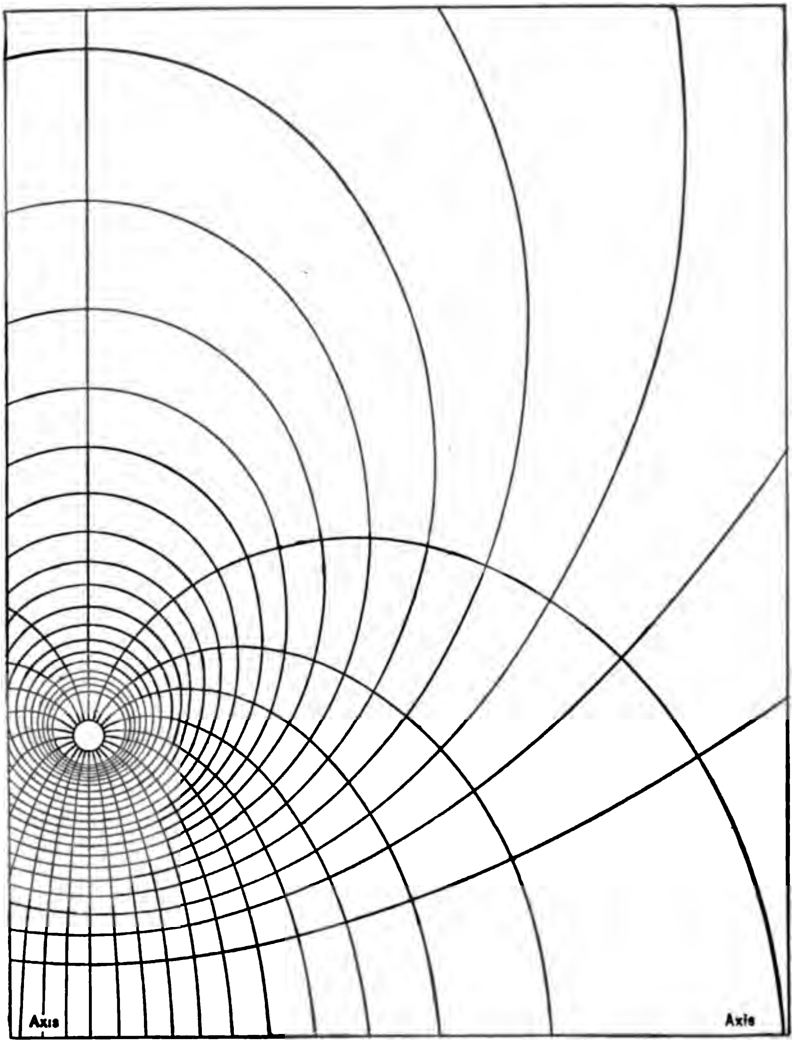


Fig. 94.

13. The Magnetic Intensity at a Point on the Axis of a Circular Linear Conductor Carrying a Current I. From (12), or, more

readily, from the direct application of the third law of motion to (7), § 8, the intensity at a point on the axis distant d from the center of the circle is, numerically,

$$H = F/m = IR^2/2(R^2 + d^2)^{3/2} \tag{16}$$

and is directed along the axis in the positive direction through the circuit.

If the circular circuit contains n turns, closely wound, instead of one,

$$H = nIR^2/2(R^2 + d^2)^{3/2} \tag{17}$$

and if there are two similar coils at the same distance d on opposite sides of P, with their currents of the same magnitude and in the same direction,

$$H = nIR^2/(R^2 + d^2)^{3/2} \tag{18}$$

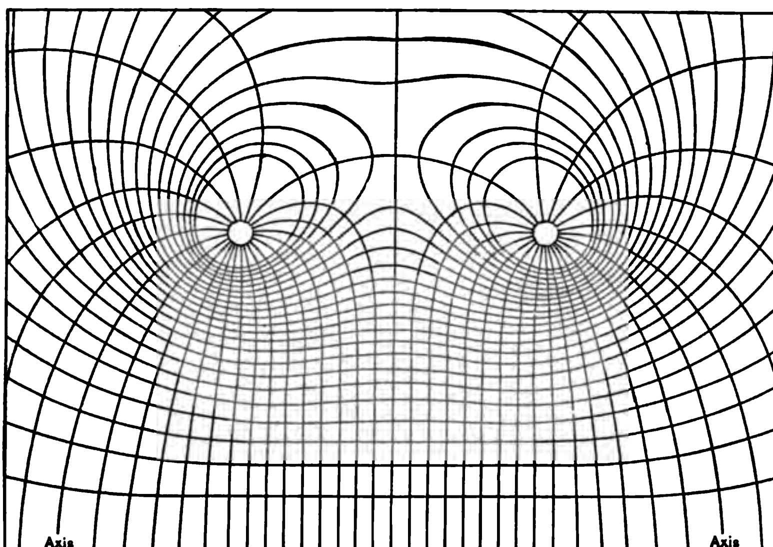


Fig. 95.

Putting $d = 0$, we obtain for the intensity at the center of a circular coil of n turns,

$$H = nI/2R \tag{19}$$

Maxwell's diagram of the field connected with a single coil is given in Fig. 94, and that of the field connected with two similar

parallel coils carrying the same current in the same direction in Fig. 95 (see Maxwell's *Treatise*, §§ 487, 702, and 713, from which the figures are taken).

14. The M.M.F. Around an Infinite Linear Straight Wire Carrying a Current I. The magnetomotive force along the arc P_1P_2 from P_1 to P_2, two points on the same line of intensity of radius d, Fig. 96, is

$$H \times \text{arc } P_1P_2 = I \text{ arc } P_1P_2/2\pi d = I\,\theta/2\pi \tag{20}$$

where θ denotes the angle subtended at the wire by the arc P_1P_2.

The equation shows that the m.m.f. along the arc of *any* line of intensity from a plane ABE to a plane CBE, intersecting in

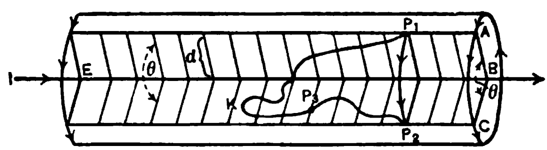


Fig. 96.

the wire at the angle θ, is equal to $I\,\theta/2\pi$, independently of the value of d.

The m.m.f., moreover, along any path not enclosing the circuit, as P_1KP_3 from P_1, any point in the plane ABE, to P_3, any point in the plane CBE, is the same, and equal to the value given in (20). For the line integration of the intensity along any path is compounded of integrations along lines of intensity, integrations along radii, and integrations parallel to the wire. The integrals in the last two directions are zero, since they are perpendicular to the intensity, and the total integral along the lines of intensity is as before, $I\,\theta/2\pi$.

The m.m.f. between two points P_1 and P_2 therefore depends only on the strength of the current I and the fraction of a circumference $\theta/2\pi$ traversed in passing from P_1 to P_2 or from P_2 to P_1.

Thus the m.m.f. along a closed circuit not enclosing the current, as the line $P_1KP_3P_2P_1$, is zero, no fraction of a circumference

being traversed, on the whole, in passing from P_1 to P_1 again. This may be shown also as follows: The m.m.f. along the path $P_1KP_3P_2$ is $I\,\theta/2\pi$, and the m.m.f. along P_2P_1 is $-I\,\theta/2\pi$. Hence the total m.m.f. is zero.

The m.m.f. around a closed path linking with the current once is

$$\Omega = I\,2\pi/2\pi = I \tag{21}$$

the directions of the m.m.f. and current being related like the rotation and translation of a right-handed screw. If the closed path links n times with the current, the m.m.f. is

$$\Omega = nI \tag{22}$$

This proposition will be generalised in Chapters XIII. and XV. It will there be shown that the m.m.f. along any closed path linking n times with any current is equal to n times the current. The relation is known as the *first law of circuitation*, and will be assumed in what follows.

15. The Magnetic Field of an Infinitely Long Cylindrical Homogeneous Conductor of Circular Cross-section. According to § 11, Chapter VIII., the current density, i, is uniform throughout the conductor and has the direction of its axis. Hence it follows from the symmetry of the conductor and equation (13) that the intensity at any point has no component in the direction of the radius or axis of the conductor. The lines of intensity are therefore circles centered on the axis in planes perpendicular thereto. The intensity at a point distant d from the axis of the conductor can now be determined by applying (21).

Outside the conductor, that is, for a circle of radius d greater than R, the radius of the conductor, we have, for the m.m.f. around the circle in the direction of the intensity,

$$\Omega = H2\pi d = I = \pi R^2 i$$

from which

$$H = I/2\pi d = R^2 i/2d \tag{23}$$

just as for a linear wire carrying the same current.

Inside the conductor, that is, for a circle of radius d less than R,

$$\Omega = H 2\pi d = i\pi d^2 = Id^2/R^2$$

and

$$H = id/2 = Id/2\pi R^2 \tag{24}$$

At the surface of the conductor, where $d = R$, (23) and (24) become identical as

$$H = iR/2 \tag{25}$$

At the axis of the conductor, where $d = 0$, (24) gives $H = 0$.

Maxwell's plane diagram of the magnetic field within and without the conductor is easily drawn by the method developed in Chapter II. The lines of intensity and equipotential lines *outside* the wire are exactly similar to the equipotential lines and lines of intensity, respectively, of the diagram of the electrostatic field of § 11, Chapter II. The development of the formulæ for drawing the lines of intensity so that the spaces between successive lines correspond to tubes of equal strength is left to the reader.

If the same current flows along a **homogeneous circular cylindrical shell of infinite length**, the field outside the conductor is the same as that given by (23), but inside the intensity gradually diminishes until it vanishes at the inner surface of the shell. Within the space inclosed by the inner surface of the shell there is no field. For if there were a field, it would be circular and perpendicular to the axis, and the intensity at the distance d from the axis would be given by (24). Since in this region $i = 0$, H is also zero.

The same method may be applied to the case of a conductor consisting of any number of coaxial circular shells, each infinite in length and homogeneous. The external field is the same as that which would surround a linear wire at the axis carrying the same current.

16. A Tore of Inductivity μ_2 Symmetrically Placed in a Circular Field in a Medium of Inductivity μ_1, etc. If a tore, or circular ring of constant cross-section, of inductivity μ_2 is placed around the cylindrical conductor or any of the conductors of §§ 14–15 im-

mersed in a medium of inductivity μ_1 with its axis coincident with that of the cylinder (or other conductor), the field external to the tore will remain unaffected, except during the introduction of the tore, and the intensity within the tore will also remain unaffected, by the principle of symmetry and equations (17) XI., and (21), but the induction at any point within the tore will be increased by its introduction in the ratio μ_2/μ_1.

Since the magnetic pressure perpendicular to the intensity and to the interface at a point P just outside the tore in medium 1 is $\frac{1}{2}\mu_1 H^2$ and the pressure just within the tore an infinitesimal distance from P, $\frac{1}{2}\mu_2 H^2$, there will be a normal mechanical force upon the interface equal, per unit area, to

$$T = \tfrac{1}{2}\mu_2 H^2 - \tfrac{1}{2}\mu_1 H^2 = \tfrac{1}{2}H^2(\mu_2 - \mu_1) \tag{26}$$

if considered positive when directed from medium 2 to medium 1. Thus if μ_2 is greater than μ_1, the volume of the tore will increase slightly on its introduction into the field.

This case and that of § 7, IV., are the limiting cases of § 9, IV.

In exactly the same way the matter within any (closed) tube of induction in a homogeneous medium of inductivity μ_1 may be replaced by a substance of inductivity μ_2. The external field, as well as the intensity within the tube, will not be disturbed; but the induction within the tube will be increased in the ratio μ_2/μ_1, and a normal traction will be developed at each point of the interface. If H denotes the intensity at any point of the interface, the traction at the point is given by (26).

17. The Magnetic Field of Two Infinitely Long Coaxial Circular Cylindrical Shells Traversed by the Same Current in Opposite Directions. Let I denote the current, and FF and GG the shells, Fig. 97.

The magnetic field in the region A is zero, as proved in a similar case in the last article.

The field in the region C outside both conductors is also zero, since a closed curve drawn around the outer shell encloses equal currents flowing in opposite directions.

In the region BB the field is circular, and the intensity at the distance d from the axis is $H = I/2\pi d$.

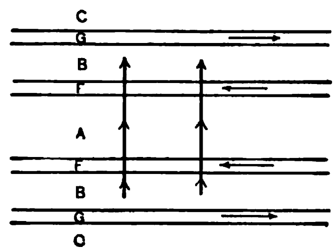


Fig. 97.

Within the shells F and G the field is also circular in the same direction, but gradually diminishes in intensity from B to C and from B to A. These intensities are computed in §§ 21 and 22, XIII.

The vanishing of the field in the region C might be regarded as due to the superposition of the field connected with FF and the field connected with GG, the intensities of the two being equal and opposite at any point of the region CC. This would also account for the gradual diminution of the intensity in passing through the outer shell from B to C.

18. The Magnetic Field of Two Parallel Circular Cylindrical Conductors carrying any currents in the same or opposite directions can be obtained at once by superposing the two fields, each already obtained in § 15. Maxwell's diagram of the lines of intensity when the wires are linear is given in Fig. 98 for the case in which the same current traverses both wires in the same direction; and the complete diagram is given in Fig. 27, II., for the case in which the same current traverses the two wires in opposite directions. When the conductors are not linear, the same diagrams hold good for the region outside the conductors, and the construction of the internal part of the diagram offers no difficulty.

When the currents are equal and opposite, the lines of intensity, as shown in the diagram, are circles about the wires. Hence the

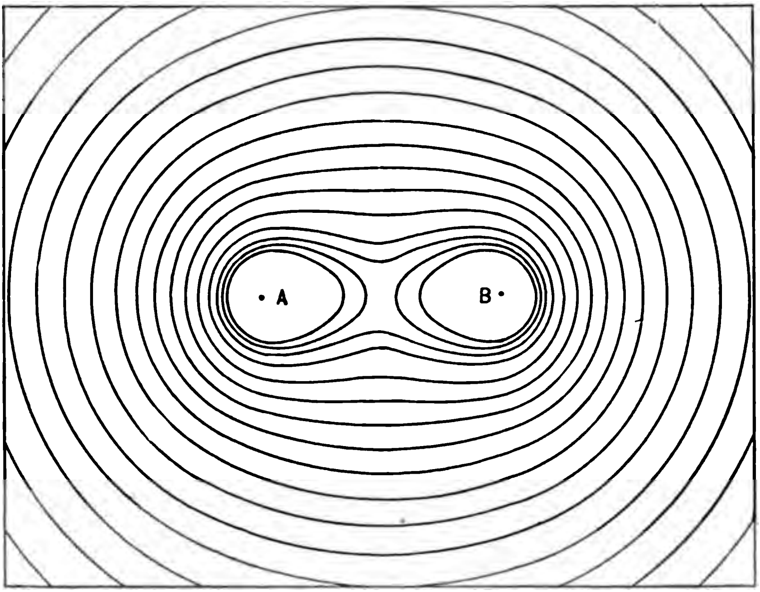


Fig. 98.

equipotential surfaces, or the equipotential lines in the diagram, are arcs of circles cutting the wires. Compare § 19, II.

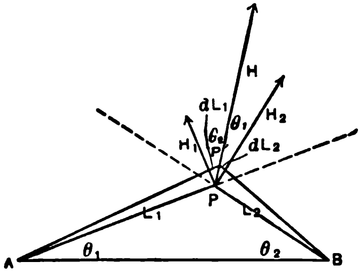


Fig. 99.

That the lines of intensity in this case are circles may be shown analytically as follows: Let the plane of the paper cut the wires perpendicularly in the points A and B, Fig. 99, and let $PP' = ds$

be the element of a line of intensity. The rest of the figure is sufficiently clear to need no further explanation. Since there is no intensity perpendicular to a line of intensity, we have at PP',

$$H_1 \cdot dL_1/ds (= \sin \theta_2) = H_2 \cdot dL_2/ds (= \sin \theta_1)$$

or

$$I/2\pi L_1 \cdot dL_1/ds = I/2\pi L_2 \cdot dL_2/ds$$

from which

$$dL_1/L_1 = dL_2/L_2$$

Integrating, we have

$$L_1/L_2 = \text{constant}$$

which is the equation of a circle with center on AB produced.

The direction of the resultant intensity H is that of the normal CPG to the equipotential (circular) arc APB through P, Fig. 100.

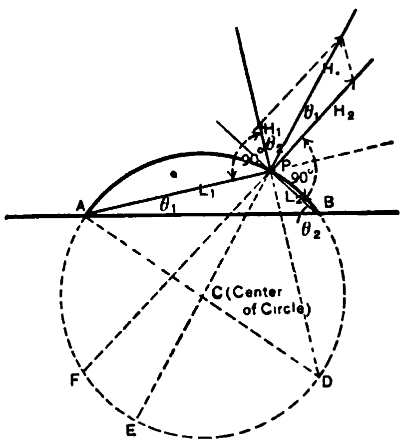


Fig. 100.

It makes with PB the angle $90° + \theta_1$, since, as is easily proved from the figure, arc PB = arc FE.

In magnitude

$$\begin{aligned} H &= H_1 \cos \theta_2 + H_2 \cos \theta_1 \\ &= I \cos \theta_2 / 2\pi AP + I \cos \theta_1 / 2\pi BP \qquad (27) \\ &= I/2\pi \cdot AB/AP \times BP = I\, AB/2\pi L_1 L_2 \end{aligned}$$

Exactly the same method used in § 19, Chapter II., might have been used in the above investigation; and the method here

used can be applied with equal facility to the problem of that article.

19. The Magnetic Field of an Infinite Solenoid. Consider an infinitely long straight coil of wire wound uniformly and closely at right angles to its length, the coil being of uniform cross-section and shape. Let there be n turns per unit length. Then, since we have supposed the wires perpendicular to the axis of the coil, the current can be regarded as forming a current sheet circulating as indicated in the figure (Fig. 101).

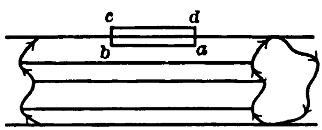


Fig. 101.

From symmetry, all the lines of intensity inside and outside the coil must be either parallel to the axis of the coil or in planes perpendicular to the axis and continuous around it. By § 11, equation (12), the second alternative is impossible, hence all the lines must be parallel to the axis.

Let H and H' denote the intensities just within and just without the coil at the point A, considered positive when in the positive direction through the coil (the direction of the current in the solenoid being chosen as the positive direction around the solenoid); and let the m.m.f. be computed for the path $abcda$, which encloses $n\ ab$ wires and therefore a current $n\ ab\ I$. Thus

$$\text{m.m.f.} = H\,ab + 0\,bc + H'cd + 0\,da = (H - H')ab = nI\,ab$$

Hence $$H - H' = nI$$

Since the m.m.f. along bc and ad is zero, the total m.m.f. around the circuit is independent of the length of bc and ad; hence $H - H'$ is constant in magnitude as well as in direction, and is equal to nI.

If S denotes the area of the coil, μHS is the magnetic flux through the coil and the return flux outside, since the tubes of

induction are continuous. Hence H' is opposite to H in direction, and since the flux returns parallel to the axis across an infinitely great area S', $\mu H' = \mu HS/S'$, or zero. Hence $H' = 0$, and

$$H = nI \tag{28}$$

everywhere within the solenoid. The intensity in the wire diminishes gradually to zero in passing from the inside to the outside of the solenoid.

Since there is no external field, the inductivity of the external medium may differ in any way from that of the internal medium without affecting the field.

In the case of an actual solenoid which is long but not infinite, the above results are obviously approximately true.

20. A. The Magnetic Field of a Finite or Infinite Circular Solenoid. The magnetic intensity at any point along the axis of the solenoid, of radius R and n turns per unit length, can be found by direct integration from equation (16). From this equation the intensity at the given point P due to the current in the infinitesimal ring of width dx distant x from P is

$$dH = nIR^2dx/2(R^2 + x^2)^{\frac{3}{2}}$$

the current in the ring being $nIdx$. The total intensity at P is therefore

$$\begin{aligned} H = \int dH &= nI/2 \int_{-L_2}^{L_1} d(x/R)/(1 + x^2/R^2)^{\frac{3}{2}} \\ &= nI/2[1/(1 + R^2/L_1^2)^{\frac{1}{2}} + 1/(1 + R^2/L_2^2)^{\frac{1}{2}}] \end{aligned} \tag{29}$$

if P is distant L_1 and L_2 from the ends of the solenoid.

If $L_2 = L_1 = L$, this equation becomes

$$H = nI/(1 + R^2/L^2)^{\frac{1}{2}} \tag{30}$$

(29) shows that if the length of the solenoid, $L_1 + L_2 = 2L$, is great in comparison with R, the intensity along the axis, and therefore (the magnetic pressure $\frac{1}{2}\mu H^2$ being remembered) the intensity throughout the volume of the solenoid, is very nearly constant and equal to nI, except near the ends of the solenoid.

In this case the external field of the solenoid is very weak except near the ends, and the external medium may be altered in any manner, except in these regions, without sensibly affecting the internal field.

If the flux through a long slender solenoid is Φ, the magnetic field at external points is very nearly the same as the field of a permanent magnet placed coincident with the solenoid and having two approximately concentrated poles, of strength $+\Phi$ and $-\Phi$, at its ends.

When $L_2 = L_1 =$ infinity, (29) reduces to (28).

B. A Very Long and Slender Cylindrical Rod of Inductivity μ Placed Within a Longer and Wider Uniformly Wound Solenoid Containing a Medium of Inductivity μ_1 Parallel to its Axis. Except near the ends of the rod, the demagnetising intensity due to its poles is small, and if the ratio of its length to its diameter is sufficiently great, negligible.

The magnetic intensity within and without the rod, except near its ends and the ends of the solenoid, is

$$H = nI$$

The magnetic induction within the rod is

$$B = \mu H$$

and that within the rest of the core

$$B_1 = \mu_1 H$$

the region near the ends being excepted.

The intensity of magnetisation of the rod, except near its ends, is

$$J = B - B_1 = (\mu - \mu_1)H$$

and the numerical strength of each of the poles, distributed over the ends of the rod, is

$$m = J\ S = (B - \mu_1 H)S = (\mu - \mu_1)HS \tag{31}$$

where S denotes the area of the cross-section of the rod.

21. The Magnetic Field Between Two Infinite Parallel Planes Traversed by Equal and Opposite Currents. The result obtained for the infinite solenoid, (19), being independent of the shape and area of the cross-section of the coil, may be applied to a coil with a rectangular cross-section of finite height and infinite width. We have then two parallel plane current sheets in

Fig. 102.

opposite directions. If the current directions are as shown in the figure (Fig. 102), and if the current per unit length of the coil perpendicular to the plane of the figure is nI, the uniform magnetic field of intensity $H = nI$ is directed perpendicularly into the plane of the paper. This result could of course have been readily obtained independently.

22. The Magnetic Field of an Endless Coil Uniformly Wound Upon a Solid of Revolution Generated by Revolving a Plane Area S about an Axis in its Plane but Not Passing Through It

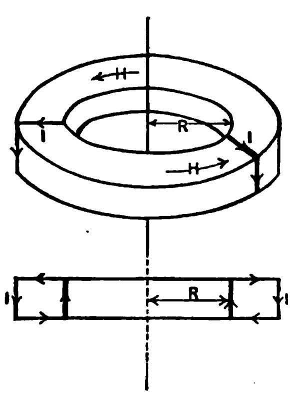


Fig. 103.

(Toroidal Coil). The current is assumed to circulate in a sheet accurately perpendicular to the coil, as shown in the figure (Fig. 103). The cross-section of the coil is shown rectangular in the figure, but may have any shape.

By symmetry, all the lines of intensity must be either circles about the axis of revolution in planes perpendicular to the axis, or closed curves around or within the coil in planes cutting it perpendicularly. The second alternative is impossible, by § 14, since such curves would enclose no current. For the same reason there can be no circular lines of intensity centered on the axis outside the coil. All such circles inside the coil, however, enclose the total current circulating around the coil. If the total number of turns of wire in the coil is N, and if the current strength is I, the m.m.f. along any circle of intensity of radius d within the coil is

$$H 2\pi d = NI$$

whence

$$H = NI/2\pi d \tag{32}$$

If the radius of the shortest line of intensity is R, and if there are n turns per unit length of this line, $N = 2\pi Rn$, and

$$H = nIR/d \tag{33}$$

If R becomes infinite, and if the area of the cross-section of the coil remains constant, so that R and d approach equality as they approach infinity,

$$H = nI$$

as otherwise shown in § 19, which with § 21, is thus a special case of the present article.

Since there is no external field, it is immaterial what medium surrounds the toroid.

23. The Force Between Two Infinite Parallel Linear Conductors Carrying Electric Currents. Let the currents, of strengths I_1 and I_2, intersect the plane of the paper at A and B respectively, distant d apart, and first suppose that the currents have the same direction, down into the plane of the paper. Then the intensity H_2 due to the current I_2 is $I_2/2\pi d$ at all points of the wire A, and is directed vertically upward at right angles to the wire. Hence the force upon a length L of the wire A is

$$F = L\mathrm{V}I_1H_2 = I_1I_2L/2\pi d \tag{34}$$

the last expression being the numerical value of the force.

$\mathrm{V}I_1H_2$ is directed toward B, thus the force is one of attraction between the wires.

The direction of F is not affected if the directions of both currents are reversed. If however the direction of one of the currents only is reversed, F becomes a force of repulsion.

The fields surrounding the wires when the currents have the same magnitude and flow in the same and opposite directions are shown in Figs. 98 and 27. The force upon each wire is seen to be always from the stronger to the weaker part of the field.

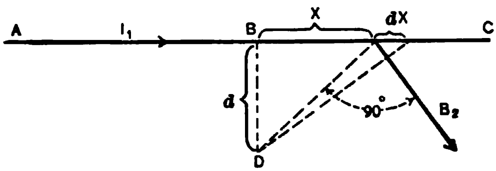


Fig. 104.

24. The Forcive Between Two Infinite Straight Linear Conductors Perpendicular to One Another and distance d apart. Let one of the conductors, AC, Fig. 104, lie in the plane of the paper, and let the other, D, be perpendicular to this plane, the currents, I_1 and I_2, being directed to the right and downward respectively. By considering the field of one current in the neighborhood of the other, the forcive is seen to be a pure torque tending to make the two currents flow in the same direction. To obtain the torque T upon a length L of AC, with center at B, we have, for the force upon an element of AC of length dx distant x from B,

$$dF = B_2I_1dx\; x/(d^2 + x^2)^{\frac{1}{2}} = \mu I_1I_2xdx/2\pi(d^2 + x^2)$$

and
$$dT = xdF = \mu I_1I_2x^2dx/2\pi(d^2 + x^2)$$

BD being chosen as axis of moments, and a positive torque tending to move C down. Hence the total torque upon a length L of AC with center at B is

$$T = \mu I_1I_2/2\pi\cdot(L + \pi d - 2d\tan^{-1} L/2d) \tag{35}$$

By considering the field of one wire in the neighborhood of the other it is easy to see that when the currents make with one another an angle θ different from 90°, there is, in addition to the torque (which vanishes when $\theta = 0°$ or 180°), a force between the wires, attractive when θ is less than 90° and repulsive when θ is greater than 90°, the force upon each element being always from the stronger to the weaker part of the resultant field.

25. The Force Between Two Linear Circular Coaxial Wires distant d apart, where d is very small in comparison with the radius of either circle. Since the field very near any thin wire is approximately the same as the field very near an infinite straight wire, the approximate force in this case is easily obtained.

If the circles have the same radius, R, and if the currents are I_1 and I_2, the force is

$$F = I_1 I_2 / 2\pi d \cdot 2\pi R = I_1 I_2 R / d \tag{36}$$

and is attractive or repulsive according as the currents flow in the same or opposite directions.

If the radii of the coils are R_1 and $R_2 = R_1 + a$, and if the distance between their planes is b,

$$F = R_1 I_1 I_2 b / (a^2 + b^2) \tag{37}$$

which is a maximum, for given values of R_1 and R_2, when $b = a$.

25. The Torque upon a Circular Cylindrical Coil of n Turns and Radius r Placed with its Center in the Axis of Two Coaxial Circular Coils each of radius R and N turns, at a point equidistant from the planes of the coils, with which its own right cross-section makes an angle θ, when R is much larger than r. The field of the larger coils will be sensibly uniform throughout the region occupied by the smaller coil and equal to its value at the center of the axis. If the distance between the planes of the two large coils is $2d$, and if the currents of the larger and smaller coils are I and i respectively, then, by § 13, if the currents in the larger coils have the same direction, the field intensity at the center of the axis is

$$H = NIR^2/(R^2 + d^2)^{\frac{3}{2}}$$

Hence, by § 10, there is a torque upon the smaller coil equal to

$$\begin{aligned} T &= ni\,\mu H \cos(90^\circ - \theta)\cdot \pi r^2 \\ &= iI\,nN\,\mu\pi r^2 R^2 \sin\theta/(R^2 + d^2)^{\frac{3}{2}} \end{aligned} \qquad (38)$$

which is a maximum when the smaller coil is perpendicular to the other two. There is no translatory force acting upon the coil as a whole.

If the same current I traverses both coils, and if the medium surrounding the coils is air or free æther (a vacuum), in which $\mu = \mu_0 = 1$,

$$T = I^2\,nN\pi r^2 R^2 \sin\theta/(R^2 + d^2)^{\frac{3}{2}} \qquad (39)$$

numerically.

From the above equations, and the equations developed in foregoing articles, for the forcive between two conductors carrying the same current, the electromagnetic unit current and the inductivity of the surrounding medium can be defined without reference to magnets of any kind.

It can easily be shown that the field at the center of the axis is most nearly uniform when d is made equal to R. In this case (39) becomes

$$T = I^2\,nN\pi r^2 \sin\theta/2^{\frac{3}{2}}R \qquad (40)$$

numerically.

27. Galvanometers. A *galvanometer* is an instrument for measuring or detecting electric currents by means of the forcive acting between a permanent magnet and a conductor traversed by an electric current. There are two general types of such instruments: In one the magnet is fixed and the conductor is movable, in the other the conductor is fixed and the magnet is movable.

28. The Deprez-D'Arsonval Galvanometer. This is an instrument of the first type. It consists essentially of a permanent horseshoe magnet NS, Fig. 105, with a very strong magnetic field between its poles; a coil C, consisting of many turns of fine

insulated wire, suspended vertically by a very fine metallic ribbon AD continuous with the wire of the coil; and, attached to the coil, a pointer or light mirror B, by which any angular change in the position of its plane may be determined.

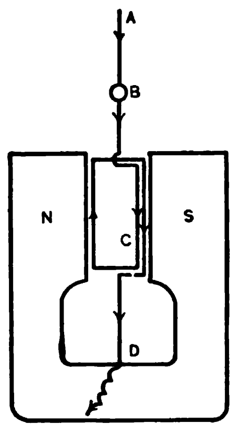


Fig. 105.

The coil is adjusted until, when no current is flowing, its plane is parallel to the magnet's field, which is so strong that the field of the earth and other magnets or currents in the neighborhood is negligible in comparison. When traversed by a current, the coil is deflected about AB as axis until the torque exerted upon it by the magnetic field is balanced by the return torque due to the twist of the elastic suspension. Let θ denote the permanent angle of deflection when the current strength is I, S and n the (average) area of a single turn of the coil and the number of turns, respectively, and $\mathbf{B}$ the magnetic induction, supposed uniform, between N and S. Then the torque upon the coil due to the field is, by (9), § 10,

$$T = nSI\mathbf{B}\cos\theta$$

If R is the torsional constant of the suspending ribbon, we have also

$$T = R\theta$$

Hence, equating the two values of T and solving for I, we have

$$I = R/nS\mathbf{B}\cdot\theta/\cos\theta \tag{41}$$

For small values of $\theta/\cos\theta$ is sensibly equal to 1, and I is sensibly proportional to θ.

By giving a special shape to the magnet's pole faces and inserting within the coil symmetrically a cylinder of steel magnetised transversely, or a cylinder of soft iron, the induction in the region moved through by the vertical portions of the coil may be made practically uniform in magnitude and parallel to the plane of the coil in any position. In this case $T = nS\mathbf{B}I = R\theta$ and

$$I = R/nS\mathbf{B}\theta \tag{42}$$

for all angles.

29. The Tangent Galvanometer. The simplest galvanometer of the second type, in which the magnet is movable and the conductor fixed, is the *tangent galvanometer*. This instrument, in its simplest form, consists essentially of a circular coil of wire, each turn having practically the same radius R, wound upon a suitable frame (of non-magnetic material); a small permanent magnet suspended by a long thread of quartz or silk as free from torsion as possible, or otherwise mounted in such a way as to move very freely, at the center of the coil; and a mirror or a pointer mounted upon the magnet, by which its angular motion may be determined. The coil is placed with its plane, or the nearly coincident planes of its turns, in the magnetic meridian, NS.

When there is no current in the coil the magnet will come to rest with its axis in this plane, under the action of the horizontal component of the earth's magnetic intensity, which will be denoted by $\mathbf{H}$. When a current I traverses the coil, there is developed at the center of the coil a magnetic intensity perpendicular to $\mathbf{H}$ and equal to

$$H = nI/2R = GI$$

$G(=H/I = n/2R)$ being a constant for the given coil.

The resultant, H', of the two fields makes with $\mathbf{H}$ an angle

$$\theta = \tan^{-1} H/\mathbf{H}$$

and the magnet will be deflected through this angle, coming to rest when its axis lies in the vertical plane through H'. Hence we have

$$I = 2R/nH = 2R\mathbf{H}/n \cdot \tan\theta = \mathbf{H}/G \cdot \tan\theta \qquad (43)$$

If the magnet is suspended at the center of the axis of two similar parallel coils carrying the same current in the same direction, we have

$$I = (R^2 + d^2)^{\frac{3}{2}}\,\mathbf{H}/nR^2 \cdot \tan\theta = \mathbf{H}/G' \cdot \tan\theta \qquad (44)$$

It can easily be shown that the field near the center of the axis is as nearly uniform as possible when d is made equal to R. In this case (44) becomes,

$$I = 2^{\frac{3}{2}} R\mathbf{H}/n \cdot \tan\theta = \mathbf{H}/G'' \cdot \tan\theta \qquad (45)$$

The Determination of a Current in Absolute Electromagnetic Measure. By measuring all the quantities in the second member of either of the above equations, a current may be determined in terms of the *REM* unit.

30. Sensitive Galvanometers (Second Type). Such a galvanometer as that just described, though valuable as a means of determining a current in absolute measure, is by no means sufficiently sensitive for most purposes for which a galvanometer is needed. The sensitiveness of a galvanometer, or the ratio of the deflection to the current, or change in deflection to change in current, may evidently be increased by increasing the number of turns and bringing them closer to the magnet, or by diminishing the effect of the external magnetic field acting upon the magnet. The latter method will be considered first.

One method of diminishing the strength of the earth's magnetic field is by placing one or more magnets, called *control magnets*, in such positions as to neutralise to a greater or less extent the earth's field. This method, while very generally practised, is open to the often serious objection that slight changes in the earth's field,

which are constantly occurring, produce very large percentage variations in the weak resultant field at the magnet, thus making the behavior of the instrument extremely inconstant.

A better method is to surround the galvanometer with a thick-walled case of iron, or several such cases, to act as a magnetic shield. This greatly reduces the field of the earth without increasing the relative prominence of its variations (§ 23, XI.; §§ 14–15, IV.).

A still better method, now always adopted for the most sensitive instruments, usually in conjunction with at least one of the other two already mentioned, is to mount on the same suspension two galvanometer magnets of as nearly as possible the same moment, with their poles turned in opposite directions. If the axes of these magnets lie accurately in the same plane, and if their moments about the axis of suspension are rigorously equal, the earth's field can exercise no directive influence upon the system. Such a magnetic system is said to be *astatic*. With this condition approximately realised in practice, the directive effect of the earth's field can be made extremely small. The current is passed around one magnet only, or else, in highly sensitive instruments, in opposite directions around the two magnets. A control magnet is used to adjust the position of the mirror as well as to regulate the sensitiveness of the instrument.

31. The Winding of a Sensitive Galvanometer of the Second Type. With respect to the winding of a sensitive galvanometer of the second type, it is evident that the magnetic intensity at a point O, the position of the magnet, Fig. 106, due to unit current in length L of wire wound upon a circular arc whose radius subtends an angle θ at O and every point of which is distant r from O, is

$$H = L \sin \theta / 4\pi r^2 \qquad (46)$$

If the same wire is wound in parallel circles with their axes through O anywhere on the surface of revolution the equation of whose generating curve is $\sin\theta / r^2 = 4\pi H/L =$ constant, the

field intensity, H, at O will be the same as that given by (46). This equation may be written

$$\sin\theta / r^2 = 1/p^2 \text{ or } r^2 = p^2 \sin\theta \qquad (47)$$

where $p^2 = \text{constant} = L/4\pi H$.

A plane section through the axis of revolution of this surface, drawn for a given value of p^2, is shown in Fig. 106.

Since all points within a surface drawn for a given value of p lie upon surfaces with smaller values of p, and therefore greater values of H; and since all points without the surface lie upon surfaces with greater values of p and smaller values of H; it follows that a given length of wire, in order that it may produce

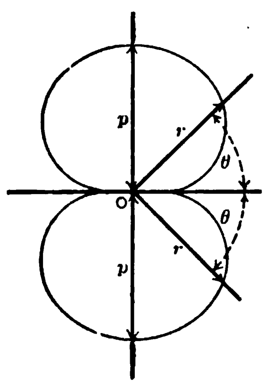


Fig. 106.

the maximum field at O, should be wound so that its whole mass just fills up the volume within the surface given by the equation (47), where the magnitude of p depends upon the quantity and the length of wire. In the actual winding, of course, some space, indicated in the figure by dotted lines, must be left for the magnet and suspension. These, however, in modern sensitive galvanometers, are so small as to weigh but a very small fraction of a gram, so that this space is not large.

Since a given length of fine wire occupies less volume than the same length of coarser wire, it is an advantage, if a galvanometer of a certain resistance is to be constructed, to use fine

wire for the smaller values of p, that is, near the magnet, and coarser wire further from the magnet. Three sizes of wire are frequently so used in the same coil, although a single wire of uniform cross-section is usually employed.

It is evident that the current traversing the coils of a galvanometer constructed as above is not, in general, strictly proportional to the tangent of the angle of deflection. The law $I = \mathbf{H}/G \cdot F(\theta)$, where $G = H/I$, connecting the current with the deflection can, however, easily be determined by experiment.

32. The Ballistic Galvanometer. The electric charge of a condenser, or the charge circulating through a coil owing to electromagnetic induction (XIII.), that is, in general, the time integral of a transitory current, can be measured with a galvanometer, provided that sensibly the whole charge can be made to circulate through the galvanometer's coil before the magnet (or coil, if the instrument is of the first type) has moved appreciably* from its position of equilibrium, and provided that the damping (or retardation due to friction, induced currents (XIII., § 5), etc.) of the magnet's (or coil's) motion is but slight. To insure that these conditions shall be satisfied, and to make the deflections of the mirror easy to read, a galvanometer designed for this purpose, called a *ballistic* galvanometer, is constructed with a magnet (or coil) of considerable moment of inertia (and long period), and damping is prevented as far as possible.

We shall consider here only an instrument of the second type. Let $\mathbf{H}$ denote the magnetic intensity at the magnet due to the earth and the control magnets, M the moment of the magnet, K its moment of inertia, H the magnetic intensity at the magnet due to a current I in the galvanometer coils, and G the constant ratio of H to I, as in § 29. Then

$$H = GI = G\, dq/dt$$

when a current $I = dq/dt$ traverses the coils.

*The cosine of the angle moved through by the magnet in the discharge time t' must not differ sensibly from unity, or a correction must be applied.

The torque upon the magnet, which remains sensibly in its position of equilibrium during the passage of the whole (appreciable part of the) charge, due to the current I is

$$T = HM = GM\, dq/dt$$

When a total charge q is passed through the coils, the (appreciable part of the) transitory current lasting for a time t' so small that the magnet does not sensibly move within this time, the total angular impulse upon the magnet is

$$\int_0^{t'} Tdt = GM\int_0^{t'} (dq/dt)dt = GM\int dq = GMq$$

If ω denotes the angular velocity imparted to the magnet by the discharge

$$\int_0^{t'} Tdt = GMq = K\omega$$

and the kinetic energy of the magnet just after the discharge, while still sensibly in its position of equilibrium, is

$$\tfrac{1}{2}K\omega^2 = \tfrac{1}{2}(GMq)^2/K$$

If, as we assume for the present, there is no damping, the magnet will come to rest at a certain angle of deflection θ, such that the work done against the magnetic field of intensity H is equal to the original kinetic energy $\tfrac{1}{2}K\omega^2$. (The torsion of the fiber suspending the magnet is supposed negligible.) Hence

$$\tfrac{1}{2}(GMq)^2/K = \int_0^{\theta} M\mathsf{H} \sin\theta d\theta = M\mathsf{H}(1 - \cos\theta)$$
$$= 2M\mathsf{H}\sin^2\tfrac{1}{2}\theta$$

from which

$$q = 2/G\cdot(K\mathsf{H}/M)^{\frac{1}{2}} \sin\tfrac{1}{2}\theta \qquad (48)$$

Thus the charge is proportional to the sine of one-half the angle of deflection (first elongation).

To obtain the charge in absolute measure from (48), it would be necessary to know both K and M, or their ratio, as well as H

and G; but the first two of these quantities can be eliminated by making use of (34), XI., according to which

$$(K/M)^{\frac{1}{2}} = T\sqrt{\mathbf{H}}/2\pi$$

T being the time in which the magnet, vibrating through an infinitesimal arc, executes a complete oscillation. Thus we have

$$q = \mathbf{H}T/\pi G \cdot \sin \tfrac{1}{2}\theta \tag{49}$$

$\mathbf{H}/G$ can be readily determined by passing a known steady current I through the coil and noting the steady deflection θ. Then, with the nomenclature of § 33, (1),

$$\mathbf{H}/G = I/F(\theta)$$

The period T of a complete vibration of the magnet for an infinitesimal arc can be computed without difficulty from the observed time of vibration for a finite arc. For small arcs the difference between the two times of vibration is not appreciable. If the damping and the torsion of the suspension are not negligible, further corrections must be applied. All these matters are discussed in Gray's *Absolute Measurements in Electricity and Magnetism.*

The development of the formula corresponding to (49) for the ballistic Deprez-D'Arsonval galvanometer is not difficult and is left to the reader. It is

$$q = \sqrt{KR}/nS\mathbf{B} \cdot \theta = R/nS\mathbf{B} \cdot T/\pi \cdot \tfrac{1}{2}\theta \tag{50}$$

The constant $R/nS\mathbf{B}$ can be easily determined from (41) by means of a known steady current and the corresponding known deflection.

In what follows (49) is adopted as a standard formula for the ballistic galvanometer. If an instrument of the first type, which offers important advantages in many cases, is used, the equations must be modified by the substitution of (50) for (49).

33. The Best Resistance for a Galvanometer Wound with Uniform Wire. We shall now determine the best resistance to be given to a set of galvanometer coils, or a galvanometer coil, of given type, in order to produce the maximum sensitiveness, when in addition to the type of galvanometer, the space to be

filled by the wire, its disposition as to the magnet, and its specific resistance are given. The volume occupied by the insulation will be supposed negligible.*

Let τ denote the volume to be occupied by the wire, and L the length, S the cross-section, r the specific resistance, and g the total resistance, of the wire. Then

$$g = rL/S = rL^2/\tau \tag{a}$$

(1) First consider a galvanometer designed to measure steady currents. The current I in the galvanometer is proportional to some function of θ, the angle of deflection, as $\tan\theta$, $\sin\theta$, $\theta/\cos\theta$, etc. Let this function be denoted by $F(\theta)$. Then we have, by what has just been said,

$$F(\theta) = BI \tag{b}$$

in which B is a constant for the given galvanometer and coil. B is evidently proportional to the length of the wire in the coil. That is

$$B = KL$$

where K is a constant depending on the size of the coils, the type of instrument, etc. (a) may therefore be written

$$F(\theta) = BI = KLI$$

or

$$I = F(\theta)/KL$$

If the galvanometer is connected in circuit with a generator with an e.m.f. Ψ and a resistance such that the total resistance in the circuit outside the galvanometer is R, we have

$$I = \Psi/(g + R) = F(\theta)/KL$$

or

$$F(\theta) = K\Psi L/(g + R) = K\Psi L/(rL^2/\tau + R)$$

* For a more complete discussion reference must be made to Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II., and to *The Physical Review*, Vol. V., p. 300.

So far as its dependence upon the length of the wire is concerned, $F(\theta)$ will be a maximum when $dF(\theta)/dL = 0$, that is when

$$dF(\theta)/dL = 0 = K\Psi(R - rL^2/\tau)/(R + rL^2/\tau)^2 = 0$$

which gives
$$rL^2/\tau = g = R \tag{51}$$

So that the greatest sensitiveness will be attained by giving the wire such a length, or such a cross-section, that its resistance is equal to the external resistance. It is obvious that r should be as small as possible to produce the maximum current with a given e.m.f. Hence the coils are usually wound of copper wire.

(2) If the galvanometer is one designed for measuring condenser charges so that the total charge q crosses every section of the wire, whatever its length or resistance, we have (§ 32)

$$F_1(\theta) = CBq = CKLq \tag{52}$$

where C is a constant and $B = KL$ can be determined by measuring $F(\theta)$, § 33, (1), when a steady current I is passed through the coils.

From (52) it is clear that the wire should be as fine as possible, or, for a given kind of wire, the resistance as great as possible. The specific resistance is immaterial, provided that the total resistance is not so great as to make the time constant (§ 41, Chapter XIII.) noticeable.

(3) If the galvanometer is to be used for measuring discharges produced by changing the magnetic flux through a coil (§ 9, Chapter XIII.), we have,

$$F_2(\theta) = CBq = CKLN/(g + R)$$

or, for a given value of N (the change of coil flux producing the discharge),

$$F_2(\theta) = \text{constant} \times L/(g + R) = \text{constant} \times L/(R + rL^2/\tau)$$

Hence $dF_2(\theta)/dL = 0$, and the sensitiveness is a maximum, when

$$g = R \tag{53}$$

as in an instrument used for steady currents. In this case also, r should be as small as possible.

34. The Electrodynamometer. The formulæ developed in § 26 are utilised for the absolute determination of current strength by the *electrodynamometer.*

The Electrodynamometer of Weber, in its simplest form, consists essentially of two large coils and a much smaller coil, similar to those described in § 26, together with a fine metallic ribbon joined to one end of the smaller coil and suspending this coil from a fixed torsion head, with its center at the center of the axis of the two large coils, a straight vertical piece of wire joined to the other end of the small coil and dipping into a cup of mercury below, and a light mirror mounted upon the small coil for reading its deflections.

The suspension is adjusted by turning the torsion head until, when there is no current, the planes of the smaller and larger coils are perpendicular. When the same current I is passed through all the coils in series, flowing in the same direction through the two larger coils, the smaller coil will be deflected through an angle θ such that the return torque due to the torsion of the suspending ribbon just balances the torque of the field. If K is the constant of torsion of the ribbon, this torque is $T = K\theta$. Hence, by § 26,

$$T = K\theta = I^2 nN\mu\pi r^2 R^2 \cos\theta / (R^2 + d^2)^{\frac{3}{2}}$$

and

$$I = K^{\frac{1}{2}}(R^2 + d^2)^{\frac{3}{4}} / rR(nN\mu\pi)^{\frac{1}{2}}(\theta / \cos\theta)^{\frac{1}{2}} \tag{54}$$

If $d = R$, when the field throughout the smaller coil is most nearly uniform,

$$I = 2^{\frac{3}{4}}(RK)^{\frac{1}{2}} / r(nN\mu\pi)^{\frac{1}{2}}(\theta / \cos\theta)^{\frac{1}{2}} \tag{55}$$

μ being sensibly equal to 1, numerically, when the coils are in air.

The Siemens Electrodynamometer. Instead of keeping the upper end of the suspension fixed, and measuring the angle of twist of the lower end, the torsion head is often turned in the di-

rection opposite to that of the deflection until the deflection is reduced to zero. If θ is the angle through which the torsion head is turned to effect this result, we have, by § 26, since the planes of the movable and fixed coils remain perpendicular,

$$T = K\theta = I^2 nN\pi r^2 R^2/(R^2 + d^2)^{\frac{3}{2}}$$

and

$$I = (R^2 + d^2)^{\frac{3}{4}}(K\theta)^{\frac{1}{2}}/(nN\mu\pi)^{\frac{1}{2}}rR \tag{56}$$

If $d = R$,

$$I = 2^{\frac{3}{4}}(KR)^{\frac{1}{2}}\theta^{\frac{1}{2}}/r(nN\mu\pi)^{\frac{1}{2}} \tag{57}$$

Since all the quantities in the second members of (54) and (56) can be determined by direct measurement, they suffice to determine I in absolute measure. Since T is proportional to the square of I, an alternating current can be measured with the electrodynamometer.

Instead of a suspension made of a single ribbon, a bifilar suspension is often used in the Weber electrodynamometer. It consists of two fine wires, parallel, or nearly parallel, connected to the two ends of the coil, serving to carry the current to and from the coil. For small angles the formulæ are the same for the two kinds of suspension.

When an electrodynamometer is not to be used for the absolute determination of current strength, but only for relative measurements, or when it is to be calibrated by comparison with standard instruments, its construction is often greatly modified to increase or diminish the sensitiveness, to reduce the size, etc.

In an electrodynamometer of the Siemens type, in which the movable coil is always in the same position when measurements are made, the current is always proportional to the square root of the angle of torsion, whatever the form of the coils.

35. The Comparison of E.M.F.s by Poggendorff's Method, Rayleigh's Modification. A constant current I is passed through the circuit $ABCDA$, Fig. 107, containing two accurately adjustable resistances R and R', by a battery D of constant e.m.f. Ψ_0 greater than any of the e.m.f.s to be compared. The agents F_1

and F_2 whose e.m.f.s Ψ_1 and Ψ_2 are to be compared are placed, one at a time, in a shunt circuit AGB, containing a galvanometer G and a key KA, in such a direction as to oppose a current through the shunt due to D. The resistances R and R' are then adjusted, while their sum $R + R'$ is kept constant and high, until the galvanometer needle remains undeflected whether the key

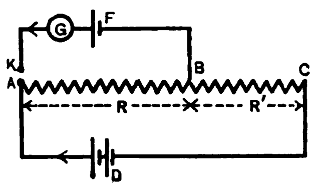


Fig. 107.

KA is open or closed. Then Ψ, the e.m.f. of the agent F, is equal to RI, the potential difference between A and B due to the field of the current. Hence, if R_1 denotes the value of R for the balance when F_1 is in the circuit, and R_2 the corresponding value when F_2 is in the circuit,

$$\Psi_1 = R_1 I \text{ and } \Psi_2 = R_2 I$$

and
$$\Psi_1/\Psi_2 = R_1/R_2 \tag{58}$$

Slight modifications of this method (potentiometer methods) enable comparisons to be made between two resistances or two currents, and are extensively used.

36. The Comparison of a Capacity and a Resistance by the Method of Direct Deflections. A constant current I traverses a circuit of high resistance including in series the resistance R under experiment. The plates of the condenser, of capacity S, are connected to the terminals of R, thus coming to a difference of potential RI and acquiring a charge SRI. The condenser is then insulated from the battery circuit and discharged through a ballistic galvanometer G, producing an angular deflection θ such that

$$SRI = \mathsf{H}T/G\pi \cdot \sin \tfrac{1}{2}\theta \tag{a}$$

The galvanometer is then disconnected from the condenser and its terminals connected to two points of the battery circuit between which the resistance is r, a very small fraction of the resistance W (which may be made as large as desired) from terminal to terminal through the galvanometer. This will not alter the current I appreciably, but will produce in the galvanometer a constant current

$$[r/(r+W)]I = \mathsf{H}/G \cdot F(\theta') \qquad (b)$$

θ' being the corresponding steady deflection. Eliminating H/GI from (a) by means of (b), we obtain

$$SR = [r/(r+W)]\,T/\pi \cdot \sin \tfrac{1}{2}\theta/F(\theta') \qquad (59)$$

The ratio $r/(r+W)$, T, θ, and $F(\theta')$ being observed, SR is given in absolute measure by (59).

The Determination of a Resistance in Absolute Measure. Since the capacity of a condenser can be calculated from its dimensions, the method affords an absolute determination of a resistance.

The Comparison of Capacities, E.M.F.s, etc., by Direct Deflections. The above disposition of apparatus, slightly modified, enables a comparison to be made between two capacities, two e.m.f.s, two resistances, or two currents.

Thus, if two condensers with capacities S_1 and S_2 are charged to the same voltage (as RI) and then discharged separately through the same ballistic galvanometer, producing angular throws of the needle equal to θ_1 and θ_2, we have, from (a),

$$S_1/S_2 = \sin \tfrac{1}{2}\theta_1/\sin \tfrac{1}{2}\theta_2 \qquad (60)$$

Or, if the same condenser is charged to different voltages V_1 and V_2 in succession, and discharged each time through the same ballistic galvanometer,

$$V_1/V_2 = \sin \tfrac{1}{2}\theta_1/\sin \tfrac{1}{2}\theta_2 \qquad (61)$$

37. The Bridge Method of Comparing Capacities. The condensers whose capacities S_1 and S_2 are to be compared are arranged as shown in Fig. 107, R_1 and R_2 being adjustable resistances. K_1 is first closed, K_2 being open. The condensers are now charged to voltages V_1 and V_2, such that $S_1V_1 = S_2V_2$, since they are connected in series. K_1 being kept closed, K_2 is then closed also. If $V_1 = R_1I$ and $V_2 = R_2I$, where I is the steady current

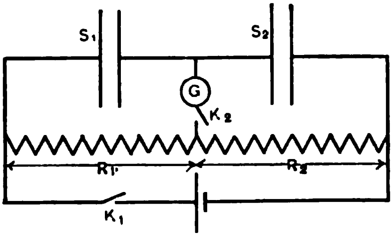


Fig. 108.

through R_1 and R_2, the needle of G will remain undeflected. Otherwise it will be deflected. In this case the condensers are discharged by opening K_1, K_2 being still closed, and the process is repeated with different ratios of R_1 to R_2 until there is no deflection on closing K_2 after K_1 (or on opening K_1 before K_2). Then

$$S_1V_1 = S_2V_2 = S_1R_1I = S_2R_2I$$

whence

$$S_1/S_2 = R_2/R_1 \tag{62}$$

The battery and galvanometer may be interchanged in the above arrangement. The ratio of the capacities is given by the same formula, (62).

CHAPTER XIII.

ELECTROMAGNETIC INDUCTION.

1. Magnetic Flux Through a Coil. Consider a thin conductor in the form of a coil of n approximately closed turns, $1, 2, \cdots, n$. Let the positive direction around any one of the turns, k, be chosen arbitrarily, and let the positive direction around each of the others be chosen to coincide with the direction of its current when the current through the coil traverses turn k in the positive direction. Let the magnetic flux through each turn be denoted by Φ with the appropriate subscript, as Φ_1, $\Phi_2, \cdots$, or Φ_n, and let Φ denote the average flux through a single turn. Then the *magnetic flux through the coil*, or the *coil flux*, which will be denoted by N, is defined by the equation

$$N = \Phi_1 + \Phi_2 + \cdots + \Phi_n = n\Phi \qquad (1)$$

If the area of the surface with its edge in any turn is not so great in comparison with the width of the conductor that the flux across the conductor itself can be neglected in comparison with the flux through the turn, an approximately correct result can be obtained by assuming the conductor (supposed circular) replaced by a *linear* conductor coinciding with its axis.

2. The Inductance, or Coefficient of Self Induction, of a coil or circuit is defined to be the quotient of the coil flux, N, due to the coil's own magnetic field divided by the current I in the coil, and will be denoted by L. Expressed in the form of an equation, this relation is

$$L = N/I \qquad (2)$$

If the inductivity μ is constant throughout the magnetic field (independent of I) so that B and N are proportional to the current

I, the inductance is the *coil flux per unit current*, and is constant. If the coil contains iron, L is far from constant.

3. Mutual Inductance or Coefficient of Mutual Induction. The *coefficient of induction* of a coil 1 with respect to a coil 2 is defined as the quotient of the coil flux N_{12} through coil 2 due to the magnetic field of coil 1 divided by the current I_1 of coil 1, and will be denoted by M_{12}. Thus

$$M_{12} = N_{12}/I_1 \tag{3}$$

which is constant, for a fixed configuration of the conductors, when μ is constant.

It will be shown later (§ 13) that when μ is constant the coefficient of induction of coil 2 with respect to coil 1 is equal to that of coil 1 with respect to coil 2. Hence we may write

$$M_{12} = M_{21} = M = N_{12}/I_1 = N_{21}/I_2 \tag{4}$$

This relation is true only when μ is constant. $M = M_{12} = M_{21}$ is called the *coefficient of mutual induction*, or the *mutual inductance*, of the two coils 1 and 2.

The coefficients of self and mutual induction will be defined from energy considerations in § 17.

4. Electromagnetic Induction. Motional Electric Intensity. It follows as a generalisation from experiment that whenever a conductor moves in a medium (æther or æther permeated by matter) supporting a magnetic field there is developed at every point P of the conductor an intrinsic electric intensity (arising from the transformation of mechanical energy into electrical or electrical into mechanical) equal to

$$e = \mathsf{V}uB \sin \gamma \tag{5}$$

where B denotes the magnetic induction at P, u the velocity of the point P of the conductor with respect to the medium (and the fixed magnetic induction at P), and γ the angle between the directions of u and B.

It is often convenient to think of the conductor as fixed and the medium supporting the magnetic field as moving, together

with the field, in the opposite direction with the velocity u at the point P. In this case we must replace (5) by

$$e = \mathrm{V}Bu \sin \gamma \tag{5a}$$

This will be done in what follows.

The electric intensity e is called a *motional* electric intensity. It is also very generally called an *induced* intensity.

A motional electric intensity in *insulators* has not yet been observed. See Blondlot, *Journal de Physique*, Jan., 1902.

Motional E.M.F. It follows from (5*a*) that the component of e at the point P in any direction is equal to the vector product

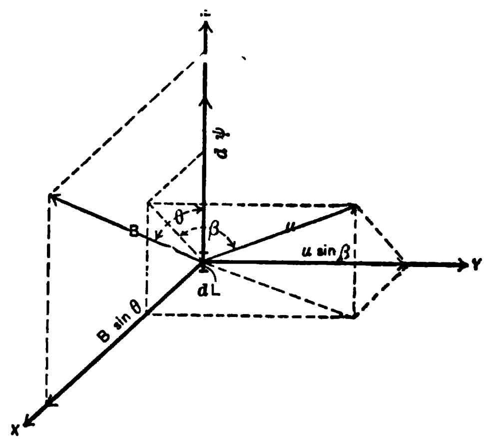


Fig. 109.

of the component of B perpendicular to this direction and the component of u perpendicular to the plane containing B and the given direction.

Hence it follows also that whenever an element, of length dL, of a line in any conductor is moving in a medium supporting a magnetic field, a *motional electromotive force* is developed along dL equal to dL multiplied by the vector product of the component of B perpendicular to dL and the component of u, the velocity of the medium (and tubes of induction) relative to dL considered as fixed, perpendicular to the plane containing B and

dL. Thus, if θ denotes the angle between B and the component of e parallel to dL, and β the angle between u and the plane containing B and the component of e just mentioned (or the element dL), the e.m.f. along dL is given in magnitude and direction by the equation

$$d\Psi = dL\mathbf{V}(B \sin \theta)(u \sin \beta) \qquad (6)$$

The relative directions of the quantities occurring in this equation are shown in Fig. 109, the axis of Z being made to coincide with the direction of dL and $d\Psi$, and the plane of XZ with the plane containing B and dL.

The second member of (6) is evidently the time rate at which magnetic flux sweeps across the element of length dL, or, numerically, the number of unit tubes of induction moving across dL per unit time. If we denote this rate, ***per unit length*** of dL, by $d\phi/dt$ (6) may be written

$$d\Psi = dL\cdot d\phi/dt \qquad (6a)$$

which, however, does not give the direction of $d\Psi$ (see *Lenz's law* below).

From the two preceding equations it is clear that the motional electromotive force along a line L of any length or form in a conductor moving in a medium supporting a magnetic field is

$$\Psi = \int dL\mathbf{V}(B \sin \theta)(u \sin \beta) = \int dL\cdot d\phi/dt = d\Phi/dt \qquad (7)$$

where $d\Phi/dt$ denotes the time rate at which magnetic flux sweeps in the same general direction across the line L, or, numerically, the number of unit tubes of magnetic induction moving across L per unit time, the last two expressions not giving the direction of Ψ. (See *Lenz's law* below.)

Motional e.m.f. and Ampère's Law. The existence of a motional e.m.f. and its origin in the transformation of mechanical work (in moving the conductor against the electromagnetic forces of § 4, XII.) into electrical energy or vice versa being established by experiment, its magnitude and direction can be deduced from Ampère's law if we *assume* that *all* the work done in moving a

conductor against electromagnetic forces is transformed into electrical energy, and that *all* the energy acquired by a moving conductor owing to the action of electromagnetic forces is transformed without loss from the energy of the electromagnetic field.

Thus suppose dL at the point P to be the element of length of a thin wire carrying a current I (due wholly or in part to the motional e.m.f.). The force upon the element, as given by Ampère's law, § 4, XII., is

$$dF = dL\mathbf{V}IB \sin \theta$$

The component of dF in the direction of the velocity u is then

$$dF' = dF \sin \beta = dL \sin \beta \; IB \sin \theta$$

and if the element is moved *against* the force dF' with the velocity $-u$ work will be done *upon* the element at the rate

$$P = dF'u = dL\,u \sin \beta \; IB \sin \theta$$

Hence the intrinsic e.m.f. developed in the length dL by the relative motion of conductor and field is

$$d\Psi = P/I = dL\,u \sin \beta \; B \sin \theta$$

in magnitude, and

$$d\Psi = dL\mathbf{V}(B \sin \theta)(u \sin \beta)$$

in both magnitude and direction, since $d\Psi$ must have the same direction as I when energy is transformed from mechanical to electrical form.

The agreement between this result and (6) justifies the above assumption.

5. Induced Electric Intensity and E.M.F. It follows from experiment that in a conductor at rest in the surrounding medium the same electric intensities (and e.m.f.s) are developed by a given motion of the tubes of magnetic induction (without motion of the medium) relatively to the conductor, due to the motion in the medium of the magnets or electric circuits producing the field, as by the same relative motion of the conductor with reference to

the fixed medium and field. This would be expected from the fact that the motional intensity can always be calculated from the motion of the tubes of induction with reference to the conductor, and the fact that the forcive upon the conductor is in both cases equal and opposite to the forcive upon the magnets or circuits producing the field, so that the work done by or against the electromagnetic forces during a given relative motion of the conductor and the magnets or circuits is in both cases the same. These intensities and e.m.f.s are not intrinsic, no energy being *transformed* from mechanical to electrical, or vice versa, *in the regions which are their seats*, since the conductor does not *move* (appreciably) against or under the action of any force. Through their agency electrical energy is *transferred* to or from the conductor at rest, the *transformation* taking place *at the circuit which moves* (in which there is a motional e.m.f.).

Also, when the magnetic flux through a circuit changes owing to any other cause than the relative *motion* of the circuit and a magnetic field, as when the current in the given circuit itself (if a conductor) or the current in a neighboring circuit *varies*, the tubes of magnetic induction must be conceived to *move* inward or outward *across* the circuit, since all tubes of magnetic induction are *continuous* or *closed*, and since each tube has the same strength throughout its length. Hence the change of the magnetic field in this manner would be expected to develop intensities and e.m.f.s similar to those developed by the relative motion of a conductor and magnets or circuits carrying steady currents; and this expectation is fully confirmed by experiment. These e.m.f.s are not intrinsic, energy being merely *transferred*, not transformed, through their agency, and all the (electrical) energy so transferred coming originally (by transformation) from the intrinsic e.m.f.s in the circuit, or one of the circuits.

The intensities and e.m.f.s considered in this section are called *induced* electric intensities and e.m.f.s, although, as stated in §4, the same term is very generally applied to the *motional* intensity

and e.m.f. also. When the intensity is not intrinsic, E should be substituted for e in (5) and (5a).

Satisfactory *direct* experiments upon the induced intensity in *insulators* have not yet been made; but the very important consequences of assuming that the results established by experiment for conductors apply to insulators as well are justified by their agreement with experiment. See Chapter XVI.

Lenz's Law. The general form of *Lenz's law*, which is only a particular case of the law of the conservation of energy, is as follows: Whenever an e.m.f. is induced in any body, either a conductor or a dielectric, by a variation of the magnetic field or by relative motion between the body and magnets or circuits traversed by electric currents, the e.m.f. has such a direction that in the resultant magnetic field the variation of the field, or the motion, which produced the e.m.f. is opposed.

Thus when a wire is moved in a magnetic field, the field is *strengthened* on the side *toward* which the wire is moving and *weakened* on the *other* side, since the force on the wire is from the stronger to the weaker part of the resultant field (§ 3, XII.). This gives the direction of the motional e.m.f. at once; for it gives the direction around the wire of the lines of intensity it produces, and this direction bears a definite relation (§ 1, XIII.) to the direction of the current developed by the motional e.m.f., which is the direction of the e.m.f. itself.

6. The Second Law of Circuitation. Integral Form. Consider any closed curve, or circuit, in a conductor or dielectric traversed by a magnetic field. Let the positive direction around the circuit be chosen arbitrarily, and the positive direction through the circuit according to the convention of § 2, XII., the first direction being related to the second as the direction of rotation to the direction of translation of a right-handed screw. Then, if the circuit moves relatively to the magnetic field in any manner, or if the field varies in any manner, or if both changes occur together, $\int dL\mathbf{V}(B \sin \theta)(u \sin \beta)$ taken *in the positive direction* once around

the circuit evidently denotes the rate at which the magnetic flux in the positive direction through the circuit is *diminishing*. If we assume that electromotive forces are induced in insulators according to precisely the same laws as in conductors, it follows that when the magnetic flux through *any* circuit changes owing to any causes an e.m.f. is induced around the circuit equal to the rate at which the magnetic flux through the circuit is decreasing. That is, if Φ denotes the magnetic flux through the circuit (positive when in the positive direction) and Ψ the e.m.f. around the circuit (positive when in the positive direction), at the time t

$$\Psi = - d\Phi / dt \tag{8}$$

which is the *second law of circuitation* (in its integral form). The e.m.f. may be wholly, or only partially, or not at all, intrinsic.

As an immediate deduction from (8) it follows that the e.m.f. induced in a coil of wire through which the coil flux changes at the rate dN/dt is

$$\Psi = - dN/dt \tag{8a}$$

The relative directions of induced electromotive force and change of magnetic flux, as given by (8) and (8*a*), can also be obtained immediately from Lenz's law.

E.M.F. of Self Induction. Thus when the current I in an isolated coil, of inductance L, increases at the rate dI/dt, and the coil flux therefore at the rate $dN/dt = d(LI)/dt$, an e.m.f. $- d(LI)dt$ is induced in the coil. Thus the change of the current or of the magnetic flux is opposed by the induced e.m.f.

E.M.F. of Mutual Induction. Also, if the current I_1 in one (1) of two coils, with mutual inductance M, increases at the rate dI_1/dt, and therefore the coil flux $N_{12} = MI_1$ through the other coil (2) due to the first coil at the rate $dN_{12}/dt = d(MI_1)/dt$, an e.m.f. $- d(MI_1)/dt$ is induced, owing to this change of flux, in coil 2 (in addition to the e.m.f. $- d(L_2I_2)/dt$). At the same time there is induced in coil 1 the e.m.f. $-[d(L_1I_1)/dt + d(MI_2)/dt]$.

7. Differential Form of the Second Law of Circuitation for Media at Rest. The Curl of a Vector. Curl E. Consider an infinitesimal plane circuit of area dS in a changing magnetic field in a medium at rest. Let B denote the induction at dS and dB its increase (in magnitude and direction) in the time dt. The line integral of *induced* electric intensity, E, around the edge of dS, *viz.*, $\int E \cos\theta\, dL$, is evidently a maximum when dS is perpendicular to dB, in which case

$$\int E \cos\theta\, dL = -d\Phi/dt = -dB/dt \cdot dS$$

Hence the maximum e.m.f. around the edge of dS per unit area is

$$\int E \cos\theta\, dL/dS = -dB/dt$$

Now the maximum line integral of a vector per unit area around the edge of an infinitesimal circuit at a point is called the *curl* of the vector at the point. Hence the above equation may be written

$$\text{curl } E = -dB/dt \tag{9}$$

Thus curl E is a vector with the magnitude and direction of $-dB/dt$. The three components of curl E along the rectangular coördinate axes of X, Y, and Z are

$$\left.\begin{aligned} \text{curl}_1 E &= -dB_1/dt \\ \text{curl}_2 E &= -dB_2/dt \\ \text{curl}_3 E &= -dB_3/dt \end{aligned}\right\} \tag{10}$$

Cartesian expressions for these components will be developed in §4, XVI.

E in the above equations denotes the (non-intrinsic) *induced* electric intensity, and does not include the motional intensity or any other intrinsic intensity, or any intensity connected with electric charges, true or fictitious, if present. Since, however, the line integral of the latter intensity around any closed circuit is zero, its curl is zero, and (9) will remain true if E is taken to denote the vector sum of the induced intensity and the intensity (whose curl is zero) due to the presence of charges, true or

fictitious. The second law of circuitation will be generalised in Chapter XV.

8. The Absolute Determination of a Resistance by Lorenz's Method. A circular coil C, Fig. 110, mounted with the planes of its turns in the magnetic meridian, is traversed by a constant current I, the resistance R to be determined being included in the circuit. With its axis coincident with that of the coil, a circular metallic disc D rotates with a constant angular velocity,

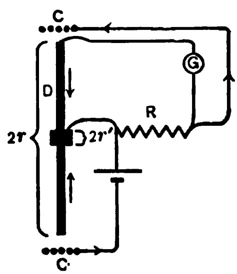


Fig. 110.

making p revolutions per second. Wires bearing upon the edge and axle of the disc connect it through a galvanometer G with the terminals of the resistance R.

Let M denote the magnetic flux across the disc between its edge and the edge of its axle per unit current in the coil. Then the magnetic flux cut across by every radius of the disc, between axle and circumference, in one complete revolution is MI (provided there is no current in the disc). Hence an e.m.f. is developed along every radius of the disc equal, from axle to circumference or from circumference to axle, to MpI. The connections from the disc to the terminals of R, for a given direction of rotation of the disc, are so arranged that the induced e.m.f. MpI and the e.m.f. RI are in opposition through the galvanometer; then the resistance or the speed of the disc is adjusted until the galvanometer shows no deflection when the galvanometer circuit is either open or closed. Then

$$MpI = RI$$

and $$R = Mp \tag{11}$$

p can be measured directly, and M can be calculated from the linear dimensions and number of turns of the coil. Hence R can be determined in absolute measure.

If C is a long solenoid, and if D is placed well within C,

$$M = n\mu \times \pi r^2 - n\mu \times \pi r'^2 = n\mu\pi(r^2 - r'^2)$$

where n denotes the number of turns of wire per cm. upon the solenoid, and r and r' denote the radii of the disc and its axle.

The method serves also for the comparison of low resistances, in which case M need not be known.

9. Induction Discharge Through a Circuit. When the coil flux through a coil of resistance R changes from one value N_1 to another value N_2 as the time changes from t_1 to t_2, an electric charge

$$q = \int Idt = \int \Psi/R \cdot dt = \int(-dN/dt)/R \cdot dt = -(N_2 - N_1)/R \tag{12}$$

due to electromagnetic induction, circulates in the positive direction around the circuit. Here I denotes that part of the current, and Ψ that part of the electromotive force, in the circuit due to the change of flux.

Thus q depends wholly upon the resistance and the total change in the flux, and not at all upon the time or the way in which this change takes place.

10. The Electrokinetic Energy of the Field of an Isolated Circuit. The magnetic energy residing in the field of a single coil or circuit whose coil flux is N when its current is I is

$$W = \tfrac{1}{2}IN = \tfrac{1}{2}LI^2 = \tfrac{1}{2}N^2/L \tag{13}$$

provided that $L = N/I =$ constant, that is, provided μ is constant.

For, if no energy is dissipated, the energy of the field is equal to the work done against the counter e.m.f. $-dN/dt$ in increas-

ing the current from 0 to I or the coil flux from 0 to N. If N and I denote also the instantaneous values of the coil flux and current, the work done against the e.m.f. $-dN/dt$, or the energy stored in the field, in the time dt is

$$dW = IdN/dt\,dt = IdN \tag{14}$$

Hence

$$W = \int_0^N IdN = \int_0^I LIdI = \tfrac{1}{2}LI^2 = \tfrac{1}{2}IN = \tfrac{1}{2}N^2/L$$

which is identical with (13).

The same result follows from § 19, XI., which gives

$$W = \tfrac{1}{2}\Phi\Omega = \tfrac{1}{2}N/n \cdot nI = \tfrac{1}{2}NI = \text{etc.}$$

W is called the *electrokinetic energy* of the field. A mechanical conception of this energy is given in § 11, B.

11. A. Mechanical Analogues of L, I, N, Ψ, dN/dt, and W. (1) Let L, I, and $N = LI$ denote the moment of inertia, angular velocity, and angular momentum, respectively, of a rigid body B about a given axis. If the angular velocity I is increased at the rate dI/dt, and the angular momentum at the rate $dN/dt = d(LI)/dt$, the increase will be opposed by a torque of inertia equal to $\Psi = -dN/dt$. To overcome this torque, that is to increase the velocity or momentum, an equal and opposite torque $+dN/dt$ must be applied. The rate at which work is done in increasing the velocity or momentum is $dW/dt = IdN/dt$, and the total work done in increasing the velocity from 0 to I, or the momentum from 0 to $N = LI$, is

$$W = \int IdN = \int LIdI = \tfrac{1}{2}IN = \tfrac{1}{2}LI^2$$

which is the kinetic energy of B when its angular velocity is I.

(2) Let L, I, and $N = LI$ denote the mass, linear velocity, and momentum, respectively, of an incompressible liquid flowing in a closed pipe of constant cross-section. If the velocity I is increased at the rate dI/dt, and the momentum at the rate

dN/dt, the increase will be opposed by a force of inertia equal to $-dN/dt = \Psi$. To overcome this inertia, that is, to increase the velocity or momentum, an equal and opposite force $+dN/dt$ must be applied. The rate at which work is done in increasing the momentum is $dW/dt = IdN/dt$, and the total work done in increasing the momentum from o to $N = LI$ is

$$W = \int IdN = \tfrac{1}{2}IN = \tfrac{1}{2}LI^2$$

which is the kinetic energy of the liquid when its velocity is I.

Neither of these analogues is at all complete. Thus the energy in (1) resides wholly in the rigid body and the energy in (2) resides wholly inside the pipe, while the energy of an electromagnetic field may reside almost wholly in the dielectric surrounding the conductor.

B. Mechanical Conception of the Magnetic Field. Let μ, H, and $B = \mu H$ denote the moment of inertia per unit volume, angular velocity, and angular momentum per unit volume, respectively, at a point P in an incompressible medium in a given type of rotatory motion. If the angular velocity H is increased at the rate dH/dt, and the angular momentum per unit volume therefore at the rate $dB/dt = \mu dH/dt$ the increase will be opposed by a torque of inertia equal, per unit volume at P, to curl $E = -dB/dt$ (E being the force per unit area acting tangential to the surface of the rotating element).

Adhering to the fundamental conception of the electric field, §§ 13–14, I., according to which E is a kind of shearing stress between positive and negative æther cells, we shall therefore assume that at any point of a magnetic field in free æther the positive and negative cells are rotating with equal angular speeds in opposite directions (unlike cells being in contact or geared together) the axes of rotation being parallel to the direction of the intensity at the point, and the direction of rotation of the positive cells being related to the direction of the intensity as the rotation to the translation of a right-handed screw. We shall assume

the magnetic intensity H to be the angular velocity of the positive cells, equal and opposite to the angular velocity of the negative cells. We shall further assume μ to be the sum per unit volume of the moments of inertia of the cells. Then $\frac{1}{2}B = \frac{1}{2}\mu H$ will be the sum of the angular momenta of the positive cells per unit volume, and $-\frac{1}{2}B = -\frac{1}{2}\mu H$ will be the sum of the angular momenta of the negative cells per unit volume. The total kinetic energy per unit volume will be $\frac{1}{2}(\frac{1}{2}\mu H \times H) + \frac{1}{2}(-\frac{1}{2}\mu H \times -H) = \frac{1}{2}\mu H^2$. The centrifugal force arising from the rotation of the cells will account for the pressure normal to the lines of magnetic intensity, and therefore for the tension along the lines of intensity, and these tensions and pressures will account for the mechanical forces upon magnets, for Ampère's law, etc. For a fuller account of this conception and its application to numerous electrical phenomena see Lodge's *Modern Views of Electricity*, and an article by W. S. Franklin, *Physical Review*, Vol. 4.

12. Electrokinetic Energy Density in a Medium With Constant Inductivity (μ). Fig. 111 represents a plane section through a

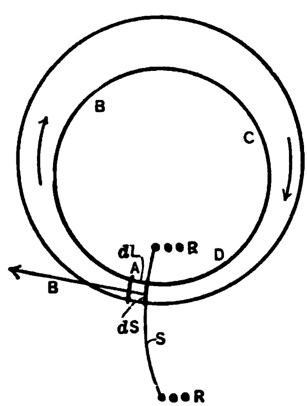


Fig. 111.

coil RR with n turns ($n = 3$ in the figure), an equipotential surface S, and a tube of induction $ABCDA$ threading the circuit RR and cutting out from the equipotential an element of area dS.

If I denotes the current through the coil, the m.m.f. around the tube $ABCDA$ is

$$nI = \int HdL$$

The magnetic flux across the equipotential S is

$$N/n = \int BdS \ .$$

Hence the energy of the magnetic field is

$$\begin{aligned} W = \tfrac{1}{2}IN = \tfrac{1}{2}nIN/n &= \tfrac{1}{2}\int HdL \int BdS \\ &= \tfrac{1}{2}\int\int BHdLdS = \int \tfrac{1}{2}BHd\tau = \int \tfrac{1}{2}\mu H^2 d\tau \end{aligned} \qquad (15)$$

the integration being extended throughout the magnetic field.

The energy per unit volume is therefore

$$T = dW/d\tau = \tfrac{1}{2}BH = \tfrac{1}{2}\mu H^2 = \tfrac{1}{2}B^2/\mu \qquad (16)$$

as otherwise proved in § 18, XI.

For the work done during the process of magnetisation when μ is a function of H see § 18, XI., or § 29.

13. The Electrokinetic Energy of the Field of Two or More Circuits in a Medium of Constant Inductivity (μ). Consider first two circuits, 1 and 2, Fig. 112, with n_1 and n_2 turns and currents I_1 and I_2, respectively. Let L_1, L_2, M_{21}, and M_{12} denote the self inductances and coefficients of induction (§ 3, XIII.) of the two circuits. Let H_1 denote the magnetic intensity at any point P due to the current I_1 alone, H_2 that due to the current I_2 alone, and H the resultant intensity due to I_1 and I_2 together. Then

$$H^2 = H_1^2 + H_2^2 + 2H_1H_2 \cos\theta_{12}$$

if θ_{12} denotes the angle between H_1 and H_2. The total energy in the magnetic field is, by § 12,

$$W = \int \tfrac{1}{2}\mu H^2 d\tau = \tfrac{1}{2}\int \mu H_1^2 d\tau + \tfrac{1}{2}\int \mu H_2^2 d\tau + \int \mu H_1 H_2 \cos\theta_{12} d\tau$$

the integrations extending throughout the magnetic field.

The first and second integrals are equal, respectively, to $\frac{1}{2}L_1I_1^2$ and $\frac{1}{2}L_2I_2^2$, by §§ 10 and 12.

To evaluate the third integral, consider a magnetic equipotential surface S for the field of I_1 alone drawn through P, and T, a tube of induction of this field enclosing P and cutting out from S

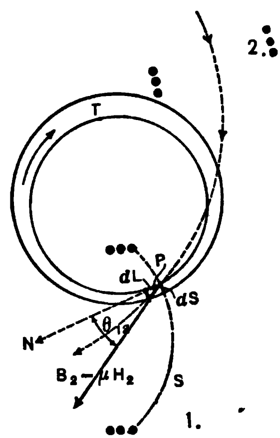


Fig. 112.

an element of area dS. With N, the normal to dS, H_2 and B_2 make the angle θ_{12}. The integral $\int \mu H_1 H_2 \cos \theta_{12} d\tau$ may evidently be written

$$\int\int \mu H_1 H_2 \cos \theta_{12} \, dS dL = \int H_1 dL \int \mu H_2 \cos \theta_{12} \, dS$$

in which the first integration extends around the tube T and the second over the surface S. Now, by (22), XII.,

$$\int H_1 dL = n_1 I_1$$

and, by the definitions of magnetic flux and coil flux,

$$\int \mu H_2 \cos \theta_{12} dS = N_{21}/n_1 = M_{21} I_2/n_1$$

Hence $\int H_1 dL \int \mu H_2 \cos \theta_{12} dS = n_1 I_1 M_{21} I_2 / n_1 = M_{21} I_2 I_1$

and $$W = \int \tfrac{1}{2} \mu H^2 d\tau = \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 + M_{21} I_2 I_1 \qquad (17)$$

If, in evaluating the third integral, we interchange the rôles of circuits 1 and 2, we obtain

$$W=\int \tfrac{1}{2}\mu H^2 d\tau = \tfrac{1}{2}L_1I_1^2+\tfrac{1}{2}L_2I_2^2+M_{12}I_1I_2 \qquad (17')$$

Since the two expressions for W are equal,

$$M_{21}=M_{12} \qquad (18)$$

as stated in § 3.

The term $M_{12}I_1I_2 = M_{21}I_1I_2$ is called the *mutual energy* of the two coils or fields. In the same way, in general, the expression $N_{21}I_1$ denotes the mutual energy of the fields 1 and 2.

In exactly the same manner it may be shown that if we have any number of circuits $1, 2, \cdots, n$ with currents $I_1, I_2, \cdots, I_n$, inductances $L_1, L_2, \cdots, L_n$, and coefficients of mutual induction $M_{12}, M_{13}, \cdots, M_{23}, M_{24}, \cdots$, etc., then the electrokinetic energy is

$$\begin{aligned}W=\tfrac{1}{2}L_1I_1^2+\tfrac{1}{2}L_2I_2^2+\cdots+\tfrac{1}{2}L_nI_n^2 \\ +M_{12}I_1I_2+M_{13}I_1I_3+\cdots+M_{23}I_2I_3+\cdots\end{aligned} \qquad (19)$$

14. The Coefficient of Self Induction of a Coil is Proportional to the Square of its Number of Turns, the dimensions of the coil being kept constant. For if Φ denotes the average flux through a single turn of the coil,

$$\Phi = knI$$

where k is a constant. Hence

$$L=N/I=n\Phi/I=kn^2 \qquad (20)$$

15. The Coefficient of Mutual Induction of Two Coils is Proportional to the Product of the Numbers of Turns in the Two Coils, the dimensions and positions of the coils remaining constant. For if Φ_{12} denotes the average flux through a single turn of coil 2 due to a current I_1 in coil 1,

$$\Phi_{12}=kn_1I_1$$

where k is a constant. Hence

$$M_{12}=N_{12}/I_1=n_2\Phi_{12}/I_1=kn_1n_2 \qquad (21)$$

16. The Coefficient of Self Induction of a Coil is Proportional to its Linear Dimensions, the shape of the coil and its number of turns remaining constant. To prove this, consider a coil A, and another coil A' with linear dimensions k times as great. If P and P' are corresponding points, S and S' corresponding equipotential surfaces, dS and dS' corresponding elements of area, B and B' corresponding inductions, Φ and Φ' corresponding average fluxes through a single turn for the same current I, and L and L' the corresponding inductances for the two coils, then $B' = 1/kB$, by § 11, XII., and $dS' = k^2 dS$. Hence

$$\Phi' = \int B' dS' = k \int B dS = k\Phi$$

and

$$L' = n\Phi'/I = kn\Phi/I = kL \tag{22}$$

which was to be proved.

In like manner it may be shown that the coefficient of mutual induction of two coils is proportional to the linear dimensions, the relative dimensions and distances retaining the same ratios.

17. Energy Definitions of Self and Mutual Inductance. From (13), § 10, the coefficient of self induction of a coil or other conductor may be defined as the ratio of twice the energy of its magnetic field to the square of its current. Thus

$$L = 2W/I^2 = \int \mu H^2 d\tau / I^2 \tag{23}$$

This definition is not identical with that of § 2 unless μ is constant. It is often more convenient than the previous definition in getting the inductance of non-linear circuits or conductors.

In like manner the coefficient of mutual induction of two circuits or conductors may be defined as the ratio of their mutual magnetic energy to the product of their currents, by § 13. Thus

$$M_{12} = (W_{12} = M_{12} I_1 I_2)/I_1 I_2 = (\int \mu H_1 H_2 \cos \theta_{12} d\tau)/I_1 I_2 \tag{24}$$

From the definitions just given the relations proved in §§ 14–16 on the basis of the earlier definitions may be readily estab-

lished. Thus if the number of turns of a coil is altered in any ratio n, the dimensions and the current remaining unaltered, the magnetic intensity H in every element of volume $d\tau$ will be altered in the same ratio. Hence $L = \int \mu H^2 d\tau / I^2$ will be altered in the square of the same ratio. The other relations may be established in like manner.

18. The Inductance of Any Number of Coils Connected in Series. If all the coils of § 13 are connected in series, $I_1 = I_2 = \cdots = I$. Hence, if L denotes the inductance of the system in series,

$$L = 2W/I^2 = L_1 + L_2 + \cdots + 2M_{12} + 2M_{13} + \cdots + 2M_{23} + \cdots \tag{25}$$

If the individual coils are so constructed or so far apart that all the mutual inductances vanish,

$$L = L_1 + L_2 + \cdots + L_n \tag{26}$$

The principle of (26) is commonly applied in the construction of standards of inductance variable by fixed amounts, and that of (25) in the construction of continuously adjustable standards of inductance. For this purpose two circular coils with inductances L_1 and L_2 are mounted (1) with their axes coincident and the distance between their centers adjustable, or (2) with their centers coincident and the angle between the planes of their turns adjustable. Thus the mutual inductance M is adjustable, and therefore the resultant self inductance,

$$L = L_1 + L_2 + 2M \tag{27}$$

M is positive or negative according as the magnetic flux due to one coil threads the other in the positive or negative direction, the direction of the current around each circuit being chosen as the positive direction around the circuit.

It will be shown in § 44, that a condenser of capacity S connected in series with a coil of inductance L produces the same effect in the case of harmonic alternating currents, as an inductance $L - 1/Sp^2$, where $p = 2\pi \times$ the frequency of the current.

19. $L_1L_2 - M^2$ is not Less than Zero. Let L_1 and L_2 denote the self inductances of two coils, and M their mutual inductance. Then

$$L_1L_2 - M^2 \geqq 0 \tag{28}$$

For the electrokinetic energy of the field, given by (17), is a signless or positive quantity, whatever the values of the currents; and (28) is the condition that this expression may never be less than zero, whatever the values of the currents.

The proposition may also be demonstrated as follows. In the nomenclature of § 15,

$$L_1 = n_1\Phi_1/I_1,\ L_2 = n_2\Phi_2/I_2, \text{ and } M = n_1\Phi_{21}/I_2 = n_2\Phi_{12}/I_1$$

Hence

$$\begin{aligned} L_1L_2 - M^2 &= n_1n_2/I_1I_2 \cdot (\Phi_1\Phi_2 - \Phi_{21}\Phi_{12}) \\ &= n_1n_2\Phi_1\Phi_2/I_1I_2 \cdot (1 - \Phi_{12}/\Phi_1 \cdot \Phi_{21}/\Phi_2) \end{aligned}$$

which is greater than zero, or equal to zero, since each of the last fractions is less than unity, or unity.

The sign of equality holds, or $L_1L_2 = M^2$, only when all the magnetic flux threads every turn of both coils ($\Phi_{12} = \Phi_{21} = \Phi_1 = \Phi_2$), as when the coils are toroidal and uniformly wound on the same core.

20. The Inductance of a Uniformly Wound Solenoid and Its Electrokinetic Energy. The flux through each turn of the infinite solenoid of § 19, XII., is $\Phi = \mu HS = \mu SnI$, and the coil flux through the nA turns in a length A of the solenoid is $N = S\mu nInA = \mu SIn^2A$. Hence the inductance of a length A of the solenoid is

$$L = N/I = \mu Sn^2A \tag{29}$$

The same result may be obtained otherwise thus: The magnetic energy per unit volume within the solenoid is $\frac{1}{2}\mu H^2 = \frac{1}{2}\mu n^2I^2$, and there is no energy outside the coil. The volume of a length A of the solenoid is SA, and the energy contained in this volume is

$$W = \tfrac{1}{2}\mu H^2 SA = \tfrac{1}{2}\mu n^2 I^2 SA \tag{30}$$

Hence
$$L = 2W/I^2 = \mu S n^2 A \tag{31}$$

Since the magnetic field is confined wholly to the region within the coil, the inductivity of the surrounding medium does not affect the inductance.

If the length, A, of a finite solenoid, § 20, XII., is great in comparison with its cross-section S, the intensity within the solenoid is sensibly uniform and equal to $H = nI$ except near the ends, where it is weaker, and the external field, except near the ends, is very weak. Hence the inductance of the solenoid is approximately that given by (31), the approximation becoming more exact as the length of the solenoid increases, since the internal energy increases almost proportionally to the length (slightly faster) and the external energy increases but slightly with increase of length (see § 8, VI.). The inductivity of the external medium affects the inductance only slightly, inappreciably when the solenoid is very long.

The permeance of the solenoid is sensibly

$$P = \Phi/\Omega = \mu HS/HA = \mu\, S/A \tag{32}$$

21. The Electrokinetic Energy Contained in an Isolated Circular Cylindrical Conductor. (1) Solid Cylinder. The energy contained in an elementary cylindrical shell of length A, radius r, and thickness dr is

$$dW = \tfrac{1}{2}\mu H^2 A 2\pi r dr$$

Within the conductor at a distance r from its axis

$$H = rI/2\pi R^2$$

if R denotes the radius of the wire and I the current. Hence

$$W = \int dW = \mu I^2 A/4\pi R^4 \int_0^R r^3 dr = \mu I^2 A/16\pi \tag{33}$$

Thus the energy within the wire, for a given current, is independent of the radius of the wire and is proportional to its induc-

tivity. The external energy is less the greater the radius of the wire.

(2) **Hollow Cylinder.** Let the inner and outer radii be denoted by R_1 and R_2, respectively. Then, within the shell, at a point distant r from the axis

$$H = I/2\pi r \cdot (r^2 - R_1^2)/(R_2^2 - R_1^2)$$

Hence $$W = \mu I^2 A/4\pi(R_2^2 - R_1^2)^2 \int_{R_1}^{R_2} (r^2 - R_1^2)^2 dr/r$$

$$= \mu I^2 A/16\pi \cdot [(R_2^2 - 3R_1^2)/(R_2^2 - R_1^2)$$

$$+ 4R_1^4/(R_2^2 - R_1^2)^2 \cdot \log R_2/R_1] \qquad (34)$$

In this case the electrokinetic energy depends on the ratio of R_2 to R_1, being greater the greater this ratio, and approaching zero as the ratio approaches unity. The correctness of the last statement is easier to see from the following considerations than from (34). For a given value of the current and R_2, the external field is wholly independent of the magnitude of R_1. The internal intensity steadily decreases from the outer to the inner surface, being equal to the external intensity at the outer surface and to zero at the inner surface. The volume of the shell approaches zero as $R_2 - R_1$ approaches zero. Hence the internal energy, which is equal to the volume of the shell times the average energy density, approaches zero as R_2/R_1 approaches unity or as $R_2 - R_1$ approaches zero, while R_1 or R_2 remains constant.

22. The Electrokinetic Energy and Inductance of a Cable consisting of a circular cylindrical core of inductivity μ and radius R and a coaxial circular cylindrical shell of the same inductivity and internal and external radii R_1 and R_2, the inductivity of the intervening dielectric being μ.

The magnetic energy contained in a length A of the core (supposed solid) is

$$(a) = \mu I^2 A/16\pi$$

The magnetic energy in a length A of the dielectric is

$$(b) = \int \tfrac{1}{2}\mu' H^2 A 2\pi r dr = \mu' I^2 A/4\pi \cdot \log R_1/R$$

The intensity within the shell at a point distant r from the axis is

$$H = I/2\pi r \cdot [1 - (r^2 - R_1^2)/(R_2^2 - R_1^2)] \tag{35}$$

Hence the magnetic energy within the shell is

$$(c) = \mu I^2 A/16\pi \cdot [(R_1^2 - 3R_2^2)/(R_2^2 - R_1^2) + 4R_2^4/(R_2^2 - R_1^2) \cdot \log R_2/R_1$$

The total magnetic energy in the length A of the cable is

$$W = (a) + (b) + (c)$$

The inductance of a length A of the system is therefore

$$L = 2W/I^2 = \mu A/8\pi + \mu' A/2\pi \cdot \log R_1/R + \mu A/8\pi \cdot [(R_1^2 - 3R_2^2)/(R_2^2 - R_1^2) + 4R_2^4/(R_2^2 - R_1^2) \cdot \log R_2/R_1] \tag{36}$$

If the outer shell is extremely thin, the third term becomes negligible, and we have, very approximately,

$$L = \mu A/8\pi + \mu' A/2\pi \cdot \log R_1/R \tag{37}$$

If the core of the cable is a very thin hollow cylinder, instead of a solid cylinder, the first term also vanishes approximately, and

$$L = \mu' A/2\pi \cdot \log R_1/R \tag{38}$$

(38) is rigorously true when the conducting shells are infinitely thin, or when the conductors are perfect conductors, both of course ideal cases.

23. The Magnetic Energy, Inductance, etc., of a Rectangular Toroid (Fig. 103, § 22, XII.). Let the uniform thickness of the toroid parallel to the axis of revolution be denoted by b, and the inner and outer radii by R_1 and R_2, and let the whole number of turns in the coil be denoted by n.

Then the m.m.f. along a closed line of intensity is

$$\Omega = nI$$

The intensity at a distance x from the axis, when x is greater than R_1 and less than R_2, is

$$H = nI/2\pi x \tag{39}$$

and the induction is

$$B = \mu H = \mu nI/2\pi x$$

The magnetic flux across an elementary strip of length b parallel to the axis, and breadth dx perpendicular to the axis is

$$d\Phi = Bbdx = \mu nbI/2\pi \cdot dx/x$$

if the strip is distant x from the axis. The total flux through a single turn of the coil is thus

$$\Phi = \mu nbI/2\pi \int dx/x = \mu nbI/2\pi \cdot \log R_2/R_1 \tag{40}$$

and the coil flux is

$$N = n\Phi = \mu n^2 bI/2\pi \cdot \log R_2/R_1$$

whence

$$L = N/I = \mu n^2 b/2\pi \cdot \log R_2/R_1 \tag{41}$$

The permeance and reluctance of the magnetic field are given by the equation

$$P = 1/R = \Phi/\Omega = \mu b/2\pi \cdot \log R_2/R_1 \tag{42}$$

The electrokinetic energy within the tore is

$$W = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\Omega\Phi = \mu n^2 bI^2/2\pi \cdot \log R_2/R_1 \tag{43}$$

24. The Inductance and Electrokinetic Energy of a System Consisting of Two Parallel Circular Cylindrical Wires traversed by the same current in opposite directions.

Let the distance d between the axes of the wires be great in comparison with R, the common radius of the wires. Then the energy contained in a length A of each wire is very nearly the same as if the wires were infinitely remote from one another, viz.,

$$(a) = \mu AI^2/16\pi$$

The energy within a length A of the dielectric is very approximately equal to $\frac{1}{2}\Phi I$, where Φ is the magnetic flux across the rectangular area $A(d-2R)$ connecting a length A of the wires.

The magnetic induction B at every point of the area $A \times (d-2R)$ is normal thereto. Let x denote the distance of a point in this plane from one of the wires, and μ' the inductivity of the dielectric; then the induction at the point is

$$B = \mu' I/2\pi x + \mu' I/2\pi(d-x)$$

Hence the magnetic flux through the area is

$$\Phi = \int_R^{d-R} B \cdot A dx = \mu' A I/\pi \cdot \log [(d-R)/R]$$

The energy contained in the tubes crossing the area $A(d-2R)$ is

$$(b) = \tfrac{1}{2}\Phi I = \mu' A I^2/2\pi \cdot \log [(d-R)/R]$$

The total energy in a length A of the system is

$$W = 2(a) + (b) = \mu A I^2/8\pi + \mu' A I^2/2\pi \cdot \log [(d-R)/R] \quad (44)$$

and the inductance of a length A is

$$L = \mu A/4\pi + \mu' A/\pi \cdot \log [(d-R)/R] \quad (45)$$

25. Two Standard Mutual Inductances. (*a*) If a coil (2) of n' turns is wound around the endless solenoid (1) of § 20, the coil flux through this outer coil (2) will be

$$N_{12} = \mu S n n' I_1$$

and the coefficient of mutual induction of the two coils will be

$$M_{12} = N_{12}/I_1 = \mu S n n' \quad (46)$$

If the solenoid has the length $2L$ and a circular cross-section of radius R, small in comparison with $2L$, the field at all points of the central portion is very nearly uniform and equal to the intensity at the center of the axis, viz.,

$$H = nI/(1 + R^2/L^2)^{\frac{1}{2}}$$

Hence if a coil of n' turns is wound about the central portion of this solenoid,

$$N_{12} = \pi R^2 \mu H n' = \frac{\pi R^2 \mu n n' I}{(1 + R^2/L^2)^{\frac{1}{2}}}$$

and the coefficient of mutual induction is

$$M_{12} = N_{12}/I = \pi R^2 \mu n n' /(1 + R^2/L^2)^{\frac{1}{2}} \qquad (47)$$

which, when L is large in comparison with R, becomes practically identical with (46), S being put equal to πR^2.

(*b*) In the same way, if a coil (2) of n' turns is wound around the closed coil (1) of § 23, the coil flux through this outer coil will be

$$N' n' = N_{12} = \mu n n' I_1 b/2\pi \cdot \log R_2/R_1$$

and the coefficient of mutual induction of the two coils will be

$$M_{12} = N_{12}/I_1 = \mu n n' b/2\pi \cdot \log R_2/R_1$$

26. The Work Done in Increasing the Coil Flux Through a Coil with a Constant Current. Let the constant current be denoted by I. If the coil flux is changing at the rate dN/dt, an e.m.f. $-dN/dt$ is developed tending to diminish the current. To keep the current constant, energy must be supplied (in addition to that supplied when the flux remains constant) to the circuit (by increasing the e.m.f. of the battery or other source) at the rate

$$dW/dt = IdN/dt$$

just sufficient to balance this induced e.m.f. The work done on the circuit, that is on the magnetic field, while the flux increases by dN during the time dt will be

$$dW/dt\, dt = dW = IdN/dt\, dt = IdN$$

Hence if the flux changes from N_1 to N_2, the magnetic field must receive an increment of energy equal to

$$W_2 - W_1 = I(N_2 - N_1) \qquad (47)$$

which is a particular case of the equation following (14).

If $N_2 - N_1$ is negative, the work $W_2 - W_1$ is also negative, or the work is done by the magnetic field, and its energy diminishes by this amount.

27. The First Law of Circuitation. The Magnetomotive Force Around a Closed Path Linking with an Electric Current. Consider an ideal permanent flexible * magnet with concentrated poles in the field of a circuit consisting of a single turn of a conductor traversed by a current which is kept constant. Suppose the negative pole of the magnet to remain fixed in position and the positive pole to be moved from its initial position along a closed path linking once with the circuit back to its initial position. During this process the flux through the circuit changes by the flux passing through the body of the magnet (equal to m, the strength of its poles), and this is the only change in the flux through the circuit that occurs. Since the pole is in its initial position, and the current is unchanged, the energy of the magnetic field is unaltered. Hence the work done by the magnetic field on the pole is equal to the work done upon the magnetic field during the change of flux. That is, if H denotes the intensity due to the current I,

$$m\int H \cos \theta dL = I\,(N_2 - N_1) = Im$$

Hence

$$\Omega = \int H \cos \theta dL = I \tag{48}$$

If there are n turns to the coil, or if the path links n times with a coil of one turn,

$$\Omega = nI$$

Thus the m.m.f. once around a closed path in the positive direction through a circuit, or in the direction of the lines of intensity, is equal to the total current in the positive direction around the circuit.

* Not essential but convenient. See Nichols and Franklin's *Elements of Physics*, Vol. II., § 124.

28. Differential Form of the First Law of Circuitation. Consider an infinitesimal circuit of area dS in a conductor traversed by a steady current whose density at dS is i. The current across dS will be a maximum, idS, when dS is turned with its normal pointing in the direction of i. In this case the m.m.f. in the positive direction around the circuit is

$$\int H \cos \theta dL = idS$$

Hence the line integral, or m.m.f., per unit area is

$$\text{curl } H = i \tag{49}$$

In the above equations H denotes the magnetic intensity of the current, and does not include the field intensity, or intrinsic intensity, connected with magnets. Inasmuch, however, as the former intensity has no curl (m.m.f. around a closed path zero), (49) will remain true if H is taken to denote the total intensity exclusive of the intrinsic intensity.

The above relation, (48) or (49), established here for *conduction* currents, is true for the general electric current (§ 7, XV.), and is called the *first law of circuitation.*

29. General Expression for the Work Done in Magnetisation. The work per unit volume done in changing the magnetic induction at any point of any medium from B_1 to B_2 was shown in § 18, XI., to be

$$dW/d\tau = \int_{B_1}^{B_2} HdB \tag{50}$$

The same result will now be deduced by a different process.

For convenience we shall make use of a very long solenoid of cross-section S uniformly wound with n turns per cm., and we shall suppose the space within the coil completely filled with a homogeneous isotropic material.

The work done in magnetising the core, in which we shall first suppose that no currents are induced during the change of magnetisation, is equal to the work done against the counter e.m.f.

developed in the coil by the changing induction. The rate at which work is thus done upon a portion of the core of length L and volume $\tau = SL$ is

$$dW/dt = IdN/dt$$

where N denotes the coil flux through the portion of the solenoid surrounding the volume considered. Hence the work done in changing the flux from N_1 to N_2, or in changing the induction from B_1 to B_2, is

$$W = \int_{N_1}^{N_2} IdN = \tau \int_{B_1}^{B_2} HdB$$

since $nI = H$ and $N = nLBS$. From this equation (50) follows immediately.

If the core is *conducting*, currents (called *eddy* currents or *Foucault* currents) will be induced in the core itself during the change of the induction, and more work than that given by (50) will be done, the extra energy going to Joulean heat. These currents and the consequent dissipation of energy can be greatly reduced by going through the process slowly, or by dividing the substance up into parts insulated from one another in the direction of flow of the eddy currents.

(50) gives the change in *magnetic energy* during the change of induction only when there is no dissipation arising from any cause.

(50) has been derived with the aid of the uniform field of a very long solenoid, but since all fields are uniform in their infinitesimal parts, the result is perfectly general.

When μ is independent of H or B, (50) reduces to (16).

30. Electrokinetic Energy, Mechanical Energy, and Change of Configuration of Circuits. First consider two circuits, 1 and 2. For the electrokinetic energy we have

$$W = \tfrac{1}{2}L_1I_1^2 + \tfrac{1}{2}L_2I_2^2 + MI_1I_2$$

If, while the currents are kept constant, the circuits so move

(or one of the circuits) that M is altered by an infinitesimal amount dM, W is altered by an amount

$$dW = I_1 I_2 dM$$

During this motion the coil flux through circuit 1 is increased by the amount

$$dN_1 = I_2 dM$$

Hence an amount of work

$$I_1 dN_1 = I_1 I_2 dM$$

is done upon the magnetic field.

In like manner, the coil flux of circuit 2 is increased by the amount

$$dN_2 = I_1 dM$$

and a second amount of work

$$I_2 dN_2 = I_2 I_1 dM$$

is done upon the magnetic field.

Hence the total work done upon the magnetic field (by the batteries, in keeping the currents constant) is

$$2 I_1 I_2 dM$$

while the increase in its energy is only

$$I_1 I_2 dM$$

Hence the additional energy $I_1 I_2 dM$ has gone to increase the mechanical energy of the system. If F denotes the force acting upon either circuit and tending to increase the distance x between the circuits, measured in its line of action, the increase of mechanical energy is

$$-Fdx = I_1 I_2 dM$$

the negative sign being chosen because dM is negative when dx is positive. Hence the force tending to increase the distance between the circuits is

$$F = -I_1 I_2 dM/dx \qquad (51)$$

If the forcive between two circuits is a torque, instead of a simple force, we have for the torque tending to increase the angle θ between the planes of the two circuits,

$$T = - I_1 I_2 dM/d\theta \tag{52}$$

31. The proposition of the last article is a particular case of the following more general theorem: If any number of circuits suffer any infinitesimal change of configuration, both L's and M's varying in the general case, while the currents are kept constant, the increase in the electrokinetic energy, dW, is equal to the increase in the mechanical energy, dW'; while the work done by the batteries (over and above that expended in overcoming resistance and dissipated in heat), dW'', is equal to

$$dW'' = dW + dW' = 2dW = 2dW' \tag{53}$$

We proceed to establish this proposition. When the most general infinitesimal change of configuration occurs, the increase in the electrokinetic energy is, when the currents are kept constant,

$$dW = \tfrac{1}{2} I_1^2 dL_1 + \tfrac{1}{2} I_2^2 dL_2 + \cdots + \tfrac{1}{2} I_n^2 dL_n$$
$$+ I_1 I_2 dM_{12} + I_1 I_3 dM_{13} + \cdots + I_h I_k dM_{hk} + \cdots$$

The work done upon the magnetic field by the batteries is

$$dW'' = I_1 dN_1 + I_2 dN_2 + \cdots + I_n dN_n$$

Now

$$dN_1 = I_1 dL_1 + I_2 dM_{12} + \cdots + I_n dM_{1n}$$

$$dN_2 = I_1 dM_{12} + I_2 dL_2 + \cdots + I_n dM_{2n}$$

$$dN_n = I_1 dM_{1n} + I_2 dM_{2n} + \cdots + I_n dL_n$$

Hence

$$dW'' = I_1(I_1 dL_1 + I_{12} dM_{12} + \cdots + I_n dM_{1n})$$
$$+ I_2(I_1 dM_{12} + I_2 dL_2 + \cdots + I_n dM_{2n})$$
$$+ \cdots + I_n(I_1 dM_{1n} + I_2 dM_{2n} + \cdots + I_n dL_n)$$

$$= d[L_1I_1^2 + L_2I_2^2 + \cdots + L_nI_n^2 + 2(M_{12}I_1I_2 + M_{13}I_1I_3 + \cdots)]$$
$$= d(2W) = 2dW$$

The difference between $dW'' = 2dW$, the energy supplied to the magnetic field, and dW, the increase in its energy, must equal the mechanical work dW' done by the field on the circuits, or the increase in the mechanical energy. Hence equation (53) immediately follows.

32. The Electrodynamic Balance. As an example of (51) we shall find the forcive upon a circular coil (2) of n turns and radius r placed with its center in the axis of a much larger circular coil (1) of N turns and radius R, the planes of the coils being parallel. Let the currents of the larger and smaller coils be denoted by I_1 and I_2 respectively, and the distance between them by d. Then

$$N_{12} = \mu H\pi r^2 n = \pi r^2 R^2 \mu n N I_1 / 2(R^2 + d^2)^{\frac{3}{2}}$$

and

$$M = N_{12}/I_1 = \pi r^2 R^2 \mu n N / 2(R^2 + d^2)^{\frac{3}{2}} \qquad (54)$$

H being practically uniform in the small region occupied by the small coil, and μ (for air) being sensibly equal to unity. The force tending to increase the distance d is, by (51),

$$F = -I_1I_2 dM/dd = -3\pi r^2 R^2 \mu n N d I_1 I_2 / 2(R^2 + d^2)^{\frac{5}{2}} \qquad (55)$$

If I_1 and I_2 have the same direction, F is attractive, otherwise repulsive. If the same current I is caused to flow through the two coils in series in the same direction, the force is

$$F = -3\pi r^2 R^2 \mu n N d I^2 / 2(R^2 + d^2)^{\frac{5}{2}} \qquad (56)$$

If a third coil C, exactly similar to the larger coil A is placed with its plane parallel to that of A and distant therefrom $2d$, with B half way between the centers of A and C, and if the same current I is made to flow through A and C in opposite directions, and also through B, the force F will be twice as great as that given by (56); or, in magnitude,

$$F = 3\pi r^2 R^2 \mu n N d I^2 / (R^2 + d^2)^{\frac{5}{2}} \qquad (57)$$

If the coils are mounted with their planes horizontal, and if the small coil B is connected with one end of the beam of a balance, the force F can be easily measured, and I determined in absolute measure. We have in this case

$$I = (R^2 + d^2)^{\frac{5}{4}} F^{\frac{1}{2}} / rR(3\pi\mu dnN)^{\frac{1}{2}} \tag{58}$$

Since F is proportional to the square of I, alternating as well as direct currents can be measured. The instrument is known as an *electrodynamic balance*.

For descriptions of two electrodynamic balances by which currents have been determined in absolute measure with great precision, see Lord Rayleigh, *Phil. Trans.*, Part II., 1884, and H. Pellat, *Comptes Rendus*, Vol. 103, 1886. The most recent of electrodynamic balances, that of von Helmholz, is described by Kahle, *Zeitschrift für Instrumentenkunde*, Vol. 17, 1897.

33. The Torsion Electrodynamometer. If the planes of the coils A and B of the last article make with one another an angle θ,

$$M = \pi r^2 R^2 nN \mu \cos\theta / 2(R^2 + d^2)^{\frac{3}{2}} \tag{59}$$

and there is, in addition to the force

$$F = -I_1 I_2 dM/d\theta = 3\pi r^2 R^2 nNd\, \mu I_1 I_2 \cos\theta / 2(R^2 + d^2)^{\frac{5}{2}} \tag{60}$$

a torque in the direction of increase of the angle θ equal to

$$T = -I_1 I_2 dM/d\theta = \pi r^2 R^2 nN I_1 I_2 \mu \sin\theta / 2(R^2 + d^2)^{\frac{3}{2}} \tag{61}$$

If the smallest coil is at the center of the two coils A and C, § 32, we have

$$F = -I_1 I_2 dM/dd = 0$$

and

$$\begin{aligned} T &= -I_1 I_2 dM/d\theta \\ &= -I_1 I_2 d/d\theta \cdot [\pi r^2 R^2 nN\mu \cos\theta/(R^2 + d^2)^{\frac{3}{2}}] \\ &= \mu I_1 I_2 \pi r^2 R^2 nN \sin\theta/(R^2 + d^2)^{\frac{3}{2}} \end{aligned} \tag{62}$$

from which the equations for the Weber and Siemens forms of electrodynamometer immediately follow, as in § 34, Chapter XII.

For a description of one of the most recent and accurate of torsion electrodynamometers see Patterson and Guthe, *Physical Review*, Vol. 7, 1898.

34. General Definitions of B and Φ. In XI. μ was defined by (2), and B by (5). These definitions are not valid, however, in their unmodified form without qualification when the magnetisation is wholly or partially intrinsic (§ 22, XI.). We shall now redefine B by (9). Thus

$$B = -\int_0^t \text{curl}\, E\, dt \tag{63}$$

the substance considered being in its neutral state at the time $t = 0$.

Consistently with the preceding definition of B and XI., we shall redefine Φ by the equation

$$\Phi = \int B \cos\theta dS = -\int_0^t \Psi dt \tag{64}$$

the substance being in its neutral state at the time $t = 0$.

Φ and B, defined by these equations, can evidently be determined experimentally (by the ballistic methods referred to below). Starting with these definitions it can be shown by experiment that tubes of magnetic induction are always closed, as assumed in § 14, XI. These definitions are perfectly consistent with the earlier definitions, and are more general, including all cases of intrinsic as well as elastic magnetisation.

35. Magnetisation Curve. Redefinition of μ. Permeability. If we start with a substance in a neutral state and increase the magnetic intensity H from zero in a series of steps, observing the corresponding values of B, we obtain, on platting the results graphically, a curve showing the relation between B and H and called the ***magnetisation curve*** of the substance.

We shall now redefine μ, for any given value of H, as the ratio of B to H for the given point on the magnetisation curve. The relation $B = \mu H$ thus holds for the process of magnetisation represented by the curve.

The curve showing the relation between $J = B(= \mu H) - \mu_1 H$ and H is also called the magnetisation curve of the substance. Either curve can of course be readily obtained from the other.

The *permeability* of a substance is the ratio of its inductivity to the inductivity of the standard medium. If, as in this book, free æther is chosen as the standard medium, the permeability μ/μ_0 of a substance is numerically equal to its inductivity. The inductivity of air is only slightly greater than unity, being equal, at ordinary temperatures and pressures, to about $\mu_0(1 + 3 \times 10^{-6})$.

36. Diamagnetic and Paramagnetic Substances. The inductivity μ of nearly every substance is independent of the value of H and is very nearly equal to $\mu_0 = 1$. Thus the magnetisation curve of such a substance is a straight line, and, if B and H are platted to the same scale, makes an angle of very nearly $45°$ with the axis of H.

A substance whose inductivity is less than μ_0, or whose permeability is less than 1, is called a *diamagnetic* substance; and a substance whose inductivity is greater than μ_0 is called a *magnetic*, a *paramagnetic*, or, if its inductivity is great and its magnetic properties resemble those of iron, a *ferromagnetic* substance. The inductivity of every diamagnetic substance is very nearly equal to unity, water being the commonest example. The inductivity of a diamagnetic or weakly magnetic substance is best investigated by methods analogous to those of §§ 3 and 4, VII. (see A. P. Wills, *Physical Review*, 6, 1898; Jäger u. Meyer, *Wied. Ann.*, 67, 1900; Du Bois, *The Magnetic Circuit*). In §§ 37 and 38 are described two methods commonly applied to iron, nickel, and cobalt, in which μ reaches great magnitudes.

For detailed information on the magnetic properties of iron and other ferromagnetic substances, together with the methods

of experimental investigation, reference may be made to Ewing's *Magnetic Induction in Iron and Other Metals* and to Du Bois's *The Magnetic Circuit.* The more important magnetic properties of iron are briefly described in Ch. Maurain's recent *Le Magnétisme du Fer.* A résumé of the magnetic properties of matter generally, with abundant references, is given by Du Bois in Vol. II. of the *Rapports* of the International Congress of Physics, 1900.

37. The Determination of B, μ, and J (Intensity of Magnetisation) by the Magnetometric Method. A very long solenoid, similar to that of § 20 B, XII., is mounted with its axis vertical, and a magnetometer (§ 26, XI.) is mounted with its needle in vertical plane passing through the solenoid perpendicular to the magnetic meridian and in a horizontal plane passing through the upper part of the solenoid. The solenoid is connected in series with another coil, called a compensating coil, so arranged that both coils together produce no deflection of the needle when traversed by a current.

The iron or other substance whose magnetisation is to be investigated, in the form of a long cylindrical rod similar to that of § 20 B, XII., is then placed vertically in the solenoid traversed by a steady current with its axis in the vertical plane perpendicular to the meridian passing through the needle, and its height is adjusted until a position is reached in which the deflection of the needle is a maximum. In this position the upper resultant pole of the rod is approximately in the horizontal plane passing through the magnetometer needle. Let $\frac{1}{2}L$ denote the distance from this plane to the center of the rod. Then the other resultant pole is, by symmetry, distant approximately $\frac{1}{2}L$ from the center on its other side. Let R denote the distance from the center of the magnetometer to the axis of the rod.

The circuit is broken, and the rod is demagnetised (if this is necessary for the purpose in view) and left, or reinserted, in the position already determined for maximum deflection. The demagnetisation can be accomplished by heating the rod to redness, or

by sending a current, gradually diminished to zero while its direction is rapidly reversed, through the solenoid with the rod in place, or by other means.

The circuit is now closed and the current increased to produce the desired value of $H = nI$ (§ 20 B, XII.), thus developing poles of strengths $+ m$ and $- m$ in the rod, and the quantities under investigation are determined as follows.

The magnetic intensity at the magnetometer needle due to the two poles of the rod is

$$H = m/4\pi\mu_1 \cdot [1/R^2 - R/(R^2 + L^2)^{\frac{3}{2}}]$$

perpendicular to the meridian.

If $\mathbf{H}$ denotes the horizontal component of the earth's magnetic field at the needle, the needle will be deflected through an angle θ such that

$$H = \mathbf{H} \tan \theta$$

By measuring θ, $\mathbf{H}$, L, R, and S, n, and the current I, the quantities B, $J = B - \mu_1 H$, $\mu - \mu_1$ can then be calculated from the above equations and those of § 20 B, XII.

If the experiments are not performed at the earth's magnetic equator, where the total intensity is horizontal, the intensity parallel to the rod will not be given completely by $H = nI$ on account of the vertical component of the earth's intensity. This component can be neutralised by winding another coil in the solenoid and passing through it a suitable current.

If after adjusting the apparatus we start with the current zero and the rod in a neutral state, and then increase the current by steps, observing both currents and corresponding deflections, we obtain, on platting the results graphically, the magnetisation curve of the substance. The magnetisation curve, and the curve showing the relation between $\mu = B/H$ and H obtained therefrom, are shown in Fig. 113 for a particular sample of wrought iron.

When great accuracy is essential, an *ellipsoid*, instead of a cylinder, of the substance under investigation must be used. See Ewing's treatise above referred to.

38. The Determination of B, μ, etc., by the Ballistic Method (Rowland's Form). The substance to be investigated, in the form of a rectangular tore, §23, whose thickness $R_2 - R_1 = a$ is small in comparison with R_1, is wound with a toroidal coil of n turns.

This coil is connected in circuit through a reversing key with a battery of constant e.m.f., a current meter, and a rheostat whose resistance can be suddenly varied.

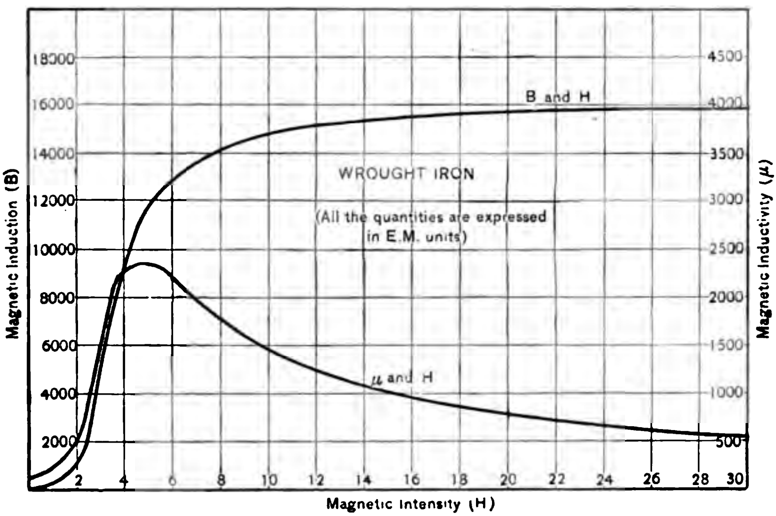


Fig. 113.

A secondary coil of n' turns is wound around the toroidal coil and connected in circuit with the coil of a ballistic galvanometer. Let the resistance of this secondary circuit be denoted by R.

The relation between the deflection of the galvanometer and the charge q circulated through its coils is supposed known from direct experiment.

Since $R_2 - R_1 = a$ is small in comparison with R_1, the magnetic intensity, given accurately by (39), is nearly constant throughout the tore and sensibly equal to

$$H = nI/\pi(R_1 + R_2)$$

The magnetic flux across every section of the tore is Bab, and the coil flux through the secondary is $n'Bab$.

If, by altering the resistance of the circuit, the current is suddenly changed from a value I_0 to a value I, the magnetic intensity will be increased by

$$H - H_0 = n(I - I_0)/\pi(R_1 + R_2) \tag{65}$$

and the magnetic induction will be increased by

$$B - B_0 = qR/n'ab \tag{66}$$

where q is the charge sent through the galvanometer circuit when the coil flux through the secondary changes from $n'B_0ab$ to $n'Bab$ (§ 9), and is known from the observed galvanometer throw.

All the quantities in the second members of (65) and (66) being determined by experiment, $H - H_0$ and $B - B_0$ are known.

Starting with any value of H_0, and the corresponding value of B_0, and increasing or decreasing the current suddenly in a series of steps, current and galvanometer throw being read at each step, the relation between $B - B_0$ and $H - H_0$ can thus be obtained for as wide a range of $H - H_0$ as desired.

If we start with the current zero, and the substance under investigation in a neutral state, H_0 and B_0 are zero, and if we increase H in a series of steps, we get the magnetisation curve of the substance. From corresponding values of B and H on this curve μ can be found by division.

For additional ballistic and other methods, see the references given above. See Du Bois, *Zeitschrift für Instrumentenkunde*, Vol. 20, 1900, for a description of his latest magnetic balance and its theory.

39. Magnetic Hysteresis. If a sample of iron, nickel, or cobalt is carried repeatedly through a given magnetising cycle, the intensity being increased to GB, Fig. 114, then reversed to $HE = - GB$, then reversed to GB, then reversed again to HE,

and so on for a number of cycles, and the relation between B and H then investigated for a complete cycle by the magnetometric or ballistic method, a closed symmetrical curve with general characteristics similar to those of the figure will result. The arrow heads indicate the direction in which the cycle is traversed.

Thus the magnetisation is in part *intrinsic.* The induction OC or OF is called the *residual* or *remanent* induction, and is the maximum value of the intrinsic induction. The ratio of the remanent induction OC or OF to the maximum induction OG or

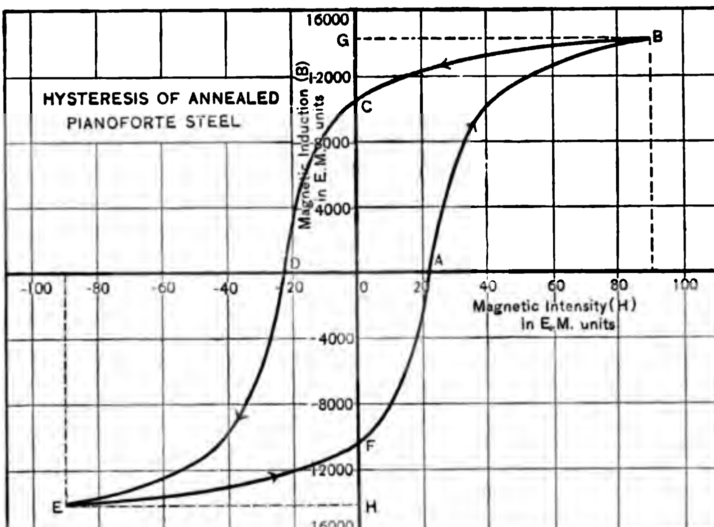


Fig. 114.

OH is called the *retentiveness* of the substance for the given cycle. The reversed intensity OD or OA necessary to reduce the intrinsic induction to zero is called the *coercive force* or *coercive intensity* of the substance for the given cycle.

The closed curve is called a *hysteresis* cycle since, as it is described, the induction always *lags* behind the intensity.

The area of the curve, viz., $\int HdB$ from any point such as B around the cycle once to the same point again, represents the work per unit volume done in carrying the substance from the

state denoted by B (or the state denoted by any other point on the curve) through the complete magnetising and demagnetising processes indicated by the cycle to the same state again. Hence $\int HdB$ for the complete cycle, or the area of the cycle, represents the energy dissipated in heat per unit volume per cycle by hysteresis.

The area of the cycle is the same or very nearly the same when the cycle is traversed very slowly and when it is traversed very rapidly (see references given below and *Comptes Rendus*, April 20, 1903). Thus the phenomenon of magnetic hysteresis is not due, except perhaps to a slight extent, to anything akin to viscosity (Cf. § 2, VI.).

For references to the literature on magnetie hysteresis, see Chapter IV. of Maurain's *Le Magnétisme du Fer* and the résumé by Warburg in Vol. II. of the *Rapports* presented to the International Congress of Physics, 1900.

If we assume the relation $B = \mu H$ to hold for the hysteresis cycle, as it holds, by definition of μ, for the magnetisation curve, μ goes through all values from $+\infty$ at C to $-\infty$ at F during the cyclic process. By introducing the ***intrinsic intensity***, $\boldsymbol{h}$, however, always acting in the direction of the induction, and by writing H for the vector sum of h and the field intensity H', which we have hitherto denoted by H, we may so define h that the relation $B = \mu H = \mu(h + H')$ (vector sum) holds universally, and leads to no impossible values of μ (cf. § 4, VI.).

The magnetic phenomena of iron, nickel, cobalt, etc., including the trend of the magnetisation curves, hysteresis, the relation of the magnetic phenomena to temperature, etc., have been largely explained by the molecular theory developed by Weber, Maxwell, and Ewing. For an extended treatment of the molecular theory and a discussion of its experimental confirmation reference must be made to Ewing's *Magnetic Induction in Iron and Other Metals*, Chapter XI.

40–44. The Current, etc., in an Electrical System Containing, in the General Case, Resistance, Inductance, and Capacity, Immersed

in a Medium, or Media, of Constant Inductivity and Permittivity. Let a condenser AB, Fig. 115, of capacity S be connected in series with a conductor CDF whose inductance is L and whose capacity is negligible in comparison with that of the condenser, and an agent with an intrinsic e.m.f. Ψ, the total resistance of the circuit being R. Let the e.m.f. Ψ be reckoned positive when directed around the circuit in the direction CDF. The agent containing the e.m.f. is supposed to be capable of being instan-

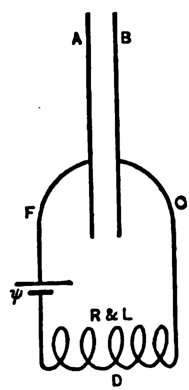


Fig. 115.

taneously inserted in or removed from the circuit, the resistance being kept constant, by suitable switches. Let the time, t, be reckoned from the instant at which Ψ is inserted or cut out. Let q denote the charge of the plate A, $I = dq/dt$ the current in the conductor in the direction CDF, V the voltage from A to B, at the time t; and let q_0, I_0, and V_0 denote the initial values of q, I, and V.

At the time t the electric energy of the system is

$$W = \tfrac{1}{2}q^2/S = \tfrac{1}{2}SV^2$$

the electrokinetic energy is

$$T = \tfrac{1}{2}LI^2 = \tfrac{1}{2}L(dq/dt)^2$$

the rate of dissipation of energy in heat is

$$F = RI^2 = R(dq/dt)^2$$

and the rate at which energy is supplied to the system by the agent with the intrinsic e.m.f. is

$$P = \Psi I = \Psi dq/dt$$

41. Non-Inductive Circuit. Case I. Let $\Psi =$ constant, $L = 0$, sensibly, and therefore $T = 0$.

A. Let Ψ be suddenly cut out of the circuit, the initial charge of A, q_0, being equal to $S\Psi$. By the principle of the conversation of energy, we have at the time t

$$-d(\tfrac{1}{2}q^2/S)/dt = R(dq/dt)^2$$

or

$$R\,dq/dt + q/S = 0 \tag{66}$$

a relation which might have been written down at once from Ohm's law $[dq/dt = I = (V_B - V_A)/R = -V/R = -q/SR]$.

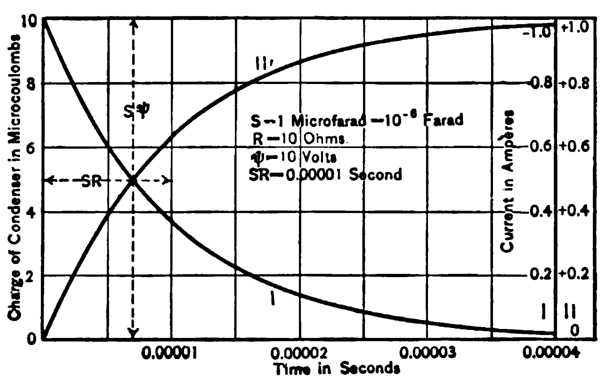


Fig. 116.

The solution of (66), with the condition $q = q_0 = S\Psi$ when $t = 0$, is

$$q = q_0 e^{-1/SR.t} = S\Psi e^{-1/SR.t} \tag{67}$$

whence

$$V = q/S = \Psi e^{-1/SR.t} = V_0 e^{-1/SR.t} \tag{68}$$

and

$$I = dq/dt = -\Psi/R \cdot e^{-1/SR.t} = I_0 e^{-1/SR.t} \tag{69}$$

The relation between q, I, and t is shown graphically for a particular case in Fig. 116, Curve I.

The time SR in which $q/q_0 = V/V_0 = I/I_0$ becomes $1/e$ is called the *time constant* of the system.

B. Let Ψ be suddenly inserted into the circuit. In this case $q_0 = 0$.

By the principle of the conservation of energy

$$\Psi\, dq/dt = R(dq/dt)^2 + d(\tfrac{1}{2}q^2/S)/dt$$

or

$$\Psi = q/S + R\, dq/dt \tag{70}$$

another equation which can be obtained immediately from Ohm's law.

To solve (70), put $q - S\Psi = q'$, and the equation becomes

$$R\, dq'/dt + q'/S = 0$$

the solution of which, with the condition $q = 0$ when $t = 0$, is

$$q' = -S\Psi e^{-1/SR.t}$$

whence

$$q = S\Psi(1 - e^{-1/SR.t}) \tag{71}$$

$$V = \Psi(1 - e^{-1/SR.t}) \tag{72}$$

$$I = \Psi/R \cdot e^{-1/SR.t} \tag{73}$$

The relations between q, I, and t are shown in Fig. 116, Curve II., for the same system whose discharge is illustrated in Curve I.

42. Case II. Inductive Circuit Without Capacity. Let Ψ = constant, S = infinity (condenser short-circuited), and therefore $W = 0$.

A. Let Ψ be suddenly cut out of the circuit, the initial value of the current being $I_0 = \Psi/R$. In this case we have, by the principle of the conservation of energy,

$$-\,d(\tfrac{1}{2}LI^2)/dt = RI^2$$

or

$$L\,dI/dt + RI = 0 \qquad (74)$$

which also follows immediately from Ohm's law $[I = (-L dI/dt)/R]$.

The solution of this equation, with the condition $I_0 = \Psi/R$, is

$$I = I_0 e^{-R/L.t} = \Psi/R \cdot e^{-R/L.t} \qquad (75)$$

The relation between I and t is shown for a particular system in Fig. 117, Scale A.

The time L/R in which I falls to $1/e$ of its initial value is called the ***time constant*** of the circuit.

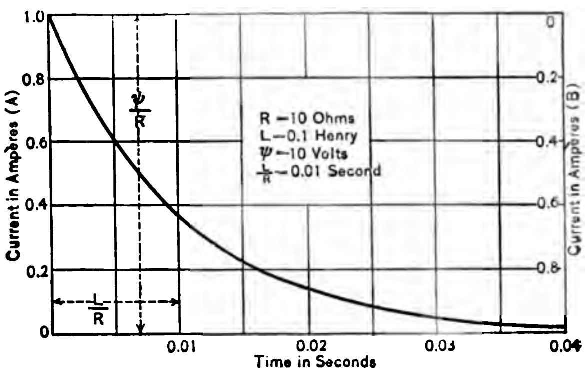


Fig. 117.

The total electric discharge in the positive direction around the circuit after the time $t = 0$ is

$$q = \int_0^\infty I dt = \Psi/R \int_0^\infty e^{-R/L.t} dt = L\Psi/R^2 = LI_0/R \qquad (76)$$

The same result follows from (12):

$$\begin{aligned} q &= -(N_2 - N_1)/R = -(0 - LI_0)/R \\ &= LI_0/R = L\Psi/R^2 \end{aligned} \qquad (77)$$

B. Let the agent with e.m.f. Ψ be suddenly inserted into the circuit, the initial value of the current being $I_0 = 0$. In this case the principle of the conservation of energy gives

$$\Psi I = RI^2 + d(\tfrac{1}{2}LI^2)/dt$$

or

$$\Psi = RI + LdI/dt \tag{78}$$

which may be obtained from Ohm's law directly $[I = (\Psi - LdI/dt)/R]$.

The solution of this equation is

$$I = \Psi/R \cdot (1 - e^{-R/L \cdot t}) \tag{79}$$

Thus I may be regarded as the sum of a steady current Ψ/R and an induced current

$$-\Psi/R \cdot e^{-R/L \cdot t}$$

The relation between I and t for the same system illustrated in Case II. is shown in Fig. 117, Scale B.

The total electric charge traversing the coil in the positive direction due to the induced current is

$$q = -L\Psi/R^2 \tag{80}$$

43. Case III. Circuit Containing Both Capacity and Inductance. Let Ψ be cut out of the circuit at the time $t = 0$. In this case the principle of the conservation of energy gives

$$-d[\tfrac{1}{2}q^2/S + \tfrac{1}{2}L(dq/dt)^2]/dt - R(dq/dt)^2$$

or

$$Ld^2q/dt^2 + Rdq/dt + q/S = 0 \tag{81}$$

which, like equations (66), (78), etc., also follows immediately from Ohm's law.

To solve (81), assume $q =$ constant $\times\ e^{mt}$ and substitute in the equation. Thus we obtain

$$Lm^2 + Rm + 1/S = 0 \tag{82}$$

The values of m which satisfy this equation are

$$m_1 = -R/2L + (R^2/4L^2 - 1/SL)^{\frac{1}{2}} = -a + (a^2 - b^2)^{\frac{1}{2}}$$

and

$$m_2 = -R/2L - (R^2/4L^2 - 1/SL)^{\frac{1}{2}} = -a - (a^2 - b^2)^{\frac{1}{2}}$$

if we put $R/2L = a$ and $1/SL = b^2$.

If m_2 is not equal to m_1, the general solution of (81) is therefore

$$q = A_1 e^{m_1 t} + A_2 e^{m_2 t} \tag{83}$$

in which A_1 and A_2 are constants to be determined by the initial conditions of the problem.

When $m_2 = m_1 = m = -a$, the solution of (81) is

$$q = (B_1 + B_2 t)e^{-at} \tag{84}$$

where B_1 and B_2 are constants to be determined by the initial conditions.

Three cases are to be considered: A, when $a^2 = b^2$; B, when $a^2 > b^2$; C, when $a^2 < b^2$.

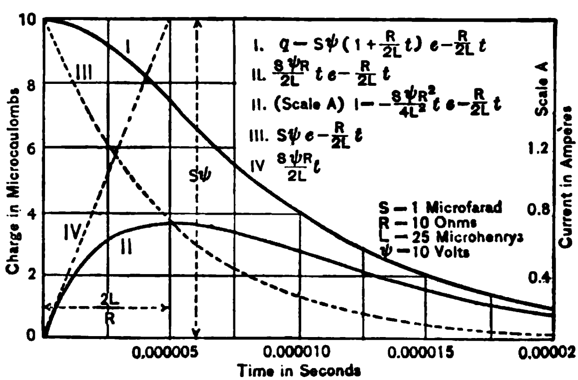


Fig. 118.

A. $a^2 = b^2$. Since $q_0 = S\Psi$ and $I_0 = 0$, (84) gives

$$q = q_0(1 + R/2L \cdot t)e^{-R/2L \cdot t} = S\Psi(1 + R/2L \cdot t)e^{-R/2L \cdot t} \tag{85}$$

$$V = \Psi(1 + R/2L \cdot t)e^{-R/2L \cdot t} \tag{86}$$

and

$$I = -S\Psi R^2/4L^2 \cdot te^{-R/2L \cdot t} \tag{87}$$

The relations between q, I, and the time for a particular system are shown in Fig. 118, I. and II. (Scale A).

B. $a^2 > b^2$. To determine the constants A_1 and A_2 we have, when $t = 0$,

$$q_0 = S\Psi = A_1 + A_2$$

and

$$I_0 = 0 = m_1 A_1 + m_2 A_2$$

Hence (83) becomes

$$q = q_0/(m_1 - m_2)\cdot(m_1 e^{m_2 t} - m_2 e^{m_1 t}) = S\Psi/(m_1 - m_2)\cdot(m_1 e^{m_2 t} - m_2 e^{m_1 t}) \qquad (88)$$

from which we obtain

$$V = q/S = \Psi/(m_1 - m_2)\cdot(m_1 e^{m_2 t} - m_2 e^{m_1 t}) \qquad (89)$$

and

$$I = dq/dt = S\Psi m_1 m_2/(m_1 - m_2)\cdot(e^{m_2 t} - e^{m_1 t}) \qquad (90)$$

The curves showing the relation between q and t and the relation between I and t are very similar to the corresponding curves of Fig. 118. Their drawing is left to the reader.

C. Oscillatory Discharge. $a^2 < b^2$. Equation (83) may be written

$$q = e^{-at}(A_1 e^{i(b^2 - a^2)^{\frac{1}{2}}t} + A_2 e^{-i(b^2 - a^2)^{\frac{1}{2}}t})$$

where i denotes $(-1)^{\frac{1}{2}}$. This equation is equivalent to

$$q = e^{-at}[A \cos (b^2 - a^2)^{\frac{1}{2}}t + B \sin (b^2 - a^2)^{\frac{1}{2}}t] \qquad (91)$$

where A and B are real constants to be determined by the initial conditions. From (91)

$$I = dq/dt = e^{-at}\{[B(b^2 - a^2)^{\frac{1}{2}} - Aa] \cos (b^2 - a^2)^{\frac{1}{2}}t - [A(b^2 - a^2)^{\frac{1}{2}} + Ba] \sin (b^2 - a^2)^{\frac{1}{2}}t\} \qquad (92)$$

From the initial conditions $q = q_0 = S\Psi$, and $I = I_0 = 0$, when $t = 0$, the above equations may be written

$$q = S\Psi e^{-at}[\cos(b^2 - a^2)^{\frac{1}{2}}t + a/(b^2 - a^2)^{\frac{1}{2}}\cdot \sin (b^2 - a^2)^{\frac{1}{2}}t] \qquad (93)$$

and

$$I = - S\Psi b^2/(b^2 - a^2)^{\frac{1}{2}}\cdot e^{-at} \sin (b^2 - a^2)^{\frac{1}{2}}t \qquad (94)$$

We have also

$$V = q/S = \text{etc.}$$

The relations between q, I, and t are shown for a particular case in Fig. 119. (See the data given in the figure.)

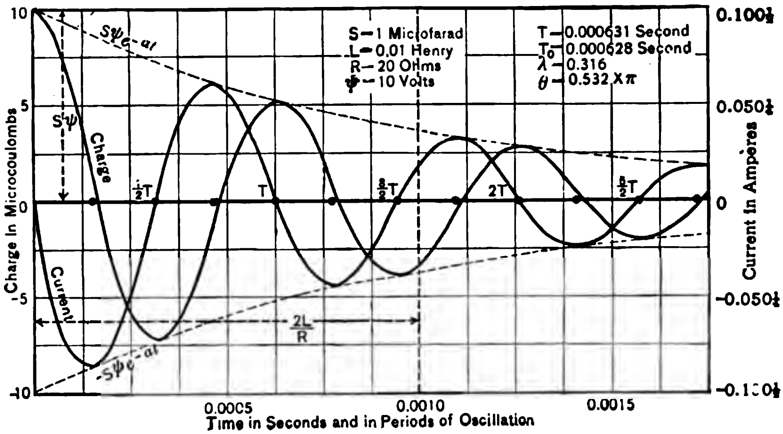


Fig. 119.

The discharge is seen to be *oscillatory*, as well as *damped*, the charge of the condenser and the current in the wire each reversing its sign at intervals of $\frac{1}{2}T$, where T is the *period* of the oscillation and is given by the equation

$$2\pi/T = (b^2 - a^2)^{\frac{1}{2}}$$

or

$$T = 2\pi/(1/SL - R^2/4L^2)^{\frac{1}{2}} \tag{95}$$

and the amplitude of the oscillation being gradually reduced to zero.

The charge q is zero at times t such that

$$t = T/2\pi \cdot [\pi - \tan^{-1}(b^2 - a^2)^{\frac{1}{2}}/a]$$

and I is zero at times t' such that

$$t' = \tfrac{1}{2}Tn$$

where n is zero or any integer.

Each quantity reaches a maximum or minimum value half way between two successive zero values.

The current reaches a maximum, minimum, or zero value ahead of the charge by the interval

$$\frac{T}{2\pi}\left(\pi - \tan^{-1}\frac{(b^2 - a^2)^{\frac{1}{2}}}{a}\right) = \frac{\theta}{2\pi}T$$

$\theta/2\pi \cdot T$ is called the *phase difference* between the current and the charge, and θ the *angle of lag* of the charge behind the current, or the *angle of lead* of the current over the charge.

The ratio of the magnitude of a maximum or minimum value of the current or charge to the magnitude of the next following minimum or maximum, occurring $\frac{1}{2}T$ later, is

$$e^{-at}/e^{-a(t+\frac{1}{2}T)} = e^{a\frac{1}{2}T}$$

The natural logarithm of this ratio is called the *logarithmic decrement* of the oscillation and is denoted by λ. Thus

$$\lambda = \tfrac{1}{2}aT = \pi/(4L/R^2S - 1)^{\frac{1}{2}} \qquad (96)$$

If $R = 0$, $a = 0$, no energy is dissipated, and the oscillation takes place without damping (energy radiated being assumed zero). The period when $R = 0$ is

$$T_0 = 2\pi(SL)^{\frac{1}{2}} \qquad (97)$$

By (95), (96), and (97) we have

$$T = T_0(1 + \lambda^2/\pi^2)^{\frac{1}{2}} = T_0(1 + \tfrac{1}{2}\lambda^2/\pi^2 + \cdots) \qquad (98)$$

Hence the effect of a small decrement on the period is proportional to the square of the decrement. See Fig. 119 for the relation between T and T_0 for the circuit whose discharge is there illustrated.

If R is increased while L and S remain constant, the period of the oscillation, as well as the damping, is increased until, when $a^2 = b^2$, the oscillatory character of the discharge disappears. As the resistance is still further increased, the discharge assumes more and more nearly the character of the discharge of a condenser through a non-inductive resistance (Case I., A).

The maximum and minimum ordinates of the current curve are equal to the corresponding (to the same time) ordinates of the logarithmic curves $+S\Psi e^{-at}$ and $-S\Psi e^{-at}$, or the logarithmic curves are tangential to the current curve at the maximum and minimum points.

The maximum and minimum ordinates of the charge curve exceed the corresponding ordinates of the logarithmic curves, which are therefore not tangential to the charge curve. The ratio of a maximum or minimum ordinate of the charge curve to the corresponding ordinate of the logarithmic curves is $b/(b^2 - a^2)^{\frac{1}{2}}$, which is nearly unity when a/b is small. For the case illustrated in the figure this ratio is about 1000 to 995, a difference scarcely perceptible in the drawing.

For one of the most recent and accurate experimental investigations confirming the above theory, which is due to Lord Kelvin (1853), the reader is referred to a memoir by Webster (*Physical Review*, 6, p. 297, 1898), where references to the earlier literature will be found.

44. Case IV. Periodic E.M.F. Let the electromotive force be a simple harmonic function of the time, $\Psi = \Psi_0 \cos pt$, where $p = 2\pi/T = 2\pi n$, T being the time of one complete period, and n the number of periods per second, or the frequency, of the electromotive force.

In this case we have, by either of the methods already frequently employed,

$$Ld^2q/dt^2 + Rdq/dt + q/S = \Psi_0 \cos pt \tag{99}$$

The general solution of this equation is the general solution of (81) already obtained, *viz.* (83), + a particular solution of (99). To obtain a particular solution of (99), assume

$$q = A \cos (pt - \theta)$$

substitute in (99), and equate to zero the coefficients of $\sin pt$ and $\cos pt$ separately (since the resulting equation must be true

for all values of t). In this way we find that the above equation is a particular solution of (99) if

$$A = \Psi_0/p[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}}$$

$$= \Psi_0/pR[1 + L/SR^2\cdot(T_0/T - T/T_0)^2]^{\frac{1}{2}} \qquad (100)$$

and

$$\theta = \tan^{-1}pR/(1/S - Lp^2)$$

Hence

$$q = \frac{\Psi_0 \cos[pt - \tan^{-1}pR/(1/S - Lp^2)]}{p[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}}} + A_1e^{m_1t} + A_2e^{m_2t} \qquad (101)$$

Since the last two terms become sensibly zero a very short time after closing the circuit, we may write, except during this interval,

$$q = \Psi_0 \cos(pt - \theta)/p[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}} \qquad (102)$$

$$I = dq/dt = \frac{\Psi_0 \cos(pt - \theta + \pi/2)}{[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}}} \qquad (103)$$

and

$$V = q/S = \Psi_0 \cos(pt - \theta)/Sp[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}} \qquad (104)$$

The angle θ is called the *phase difference* (a term more properly applied to $\theta/2\pi\cdot T$) between q and Ψ, or the *angle of lag* of q behind Ψ, or the *angle of lead* of Ψ over q. Similarly $\theta - \frac{1}{2}\pi = \theta'$ is the angle of lag of I behind Ψ, or the angle of lead of Ψ over I.

The quantity $[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}}$ is called the *impedance* of the circuit (or portion of the circuit considered), the quantity $p(L - 1/Sp^2)$ its *reactance*, and the quantity $1/Sp^2$ its *condensance*.

From the way in which S and L enter into the above equations it is evident that a condenser of capacity S in series in the circuit has the same effect upon a simple harmonic current as would a negative inductance equal to $1/Sp^2$. Thus condensance and inductance can be made to neutralise one another's effects.

For given values of Ψ and R the amplitude of I reaches a maximum when $L - 1/Sp^2 = 0$, that is when $p = p_0 = 2\pi/T_0$, where $T_0 = 2\pi(LS)^{\frac{1}{2}}$ is the natural period in which the system would execute oscillations if its resistance were zero. In this

case $\theta' = 0$, the current is *in phase* with the e.m.f., and has at all times the same value as if there were no inductance or condensance in the circuit. In this case the circuit is said to be in *resonance* with the e.m.f. The relation between the maximum value, or amplitude, pA, of the current and T/T_0 for a given value of L/SR^2 [see equation (100)] is shown in Fig. 120.

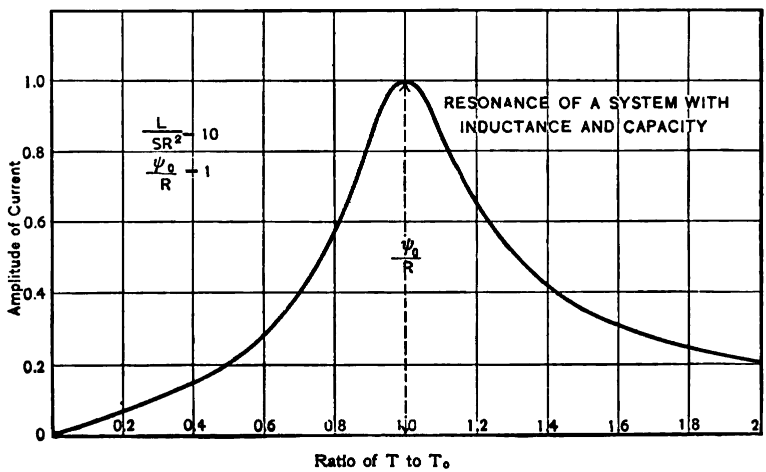


Fig. 120.

Equation (104) may be written

$$V = V_0 \cos (pt - \theta)$$

where

$$V_0 = \Psi_0 / Sp[R^2 + p^2(L - 1/Sp^2)^2]^{\frac{1}{2}} \tag{105}$$

V_0 being the amplitude of V. If the denominator in this equation is less than unity, which is easily possible, V_0 will be greater than Ψ_0, the amplitude of the applied e.m.f. Differentiation of (105) shows that for given values of S, L, and R the amplitude V_0 will be a maximum when

$$p^2 = (2L - SR^2)/2SL^2$$

If SR^2 is small in comparison with $2L$, this expression gives, approximately, $p^2 = 1/SL$, or

$$T = T_0 = 2\pi(SL)^{\frac{1}{2}}$$

the natural period of the system without damping. In this case

$$V_0/\Psi_0 = (L/SR^2)^{\frac{1}{2}}$$

which, in accordance with the above assumption ($2L/R^2S$ large), is much greater than unity. This is another illustration of the effects of *resonance*.

Equation (103) may be written

$$I = I_0 \cos (pt - \theta')$$

The power supplied to the circuit by the e.m.f. Ψ at the time t is

$$P = \Psi I = \Psi_0 I_0 \cos (pt - \theta') \cos pt \qquad (106)$$

It is easy to show, either graphically or by means of (106), that when $\theta' = 0$, P is always positive; that when θ' is equal to $\pm\pi/2$ (its limiting values in the system considered), P is positive during half the period and negative during half the period, no power on the whole being developed by the e.m.f. (to make $\theta' = \pm\pi/2$, R must be zero and either the inductance or the condensance must be zero); and that when θ' is less than $\pm\pi/2$ and greater than zero, P is positive during more than half the period.

45. Dynamical Analogues. Consider the angular motion of a cylinder C about the vertical axis of a suspending wire AB. Let the moment of inertia of C about this axis be denoted by L, and suppose the inertia of the wire negligible. When the cylinder is turned through an angle q, the twist of the wire gives rise to a return torque $V = q/S$, $1/S$ being a constant of the wire depending upon its rigidity and dimensions and called its *elastance* or the reciprocal of its permittance, tending to diminish q. Let the motion of the cylinder be resisted by a frictional torque proportional to its angular velocity $dq/dt = I$, that is by a torque $-RI = -Rdq/dt$, where R is a constant. At the time

$t = 0$, let a torque Ψ, capable of producing a final angular displacement $q_0 = S\Psi$, be applied to the system, or removed after having been applied sufficiently long to produce the maximum displacement q_0. Then the equation of motion of the cylinder after Ψ has been applied or removed may be found as follows:

The potential energy of the system is

$$W = \int V dq = \tfrac{1}{2} q^2 / S$$

the kinetic energy of the system is

$$T = \tfrac{1}{2} L I^2 = \tfrac{1}{2} L (dq/dt)^2$$

the rate of dissipation of energy in heat is

$$F = R I^2 = R (dq/dt)^2$$

the rate at which energy is supplied to the system is

$$\Psi I = \Psi dq/dt$$

By the principle of the conservation of energy

$$\Psi I = dW/dt + dT/dt + F$$

whence

$$L d^2 q/dt^2 + R dq/dt + q/S = \Psi$$

which is the equation of motion sought.

The equation may be obtained also by the direct application of the second law of motion. Thus, Ψ, the total torque acting upon the cylinder, consists of three parts: A torque $V = q/S$ to balance the counter torque due to the torsion of the wire, a torque $RI = R dq/dt$ to balance the torque $-RI$ due to friction, and a torque $L dI/dt = L d^2 q/dt^2$ to balance the torque $-L dI/dt$ due to the inertia of the cylinder, or to produce the angular acceleration dI/dt. Hence

$$\Psi = L d^2 q/dt^2 + R dq/dt + q/S$$

The equation of motion after the removal of is Ψ

$$L d^2 q/dt^2 + R dq/dt + q/S = 0$$

In the same way, if a periodic torque $\Psi = \Psi_0 \cos pt$ acts upon the cylinder, the equation of motion is

$$Ld^2q/dt^2 + Rdq/dt + q/S = \Psi_0 \cos pt$$

The solutions of these equations are given above.

I. Let the inertia of the cylinder be negligible (T/W sensibly zero). A. The cylinder is displaced initially through the angle q_0, and the wire possesses the potential energy $\frac{1}{2}q_0^2/S$. If the cylinder is suddenly released, it will begin to move with the angular velocity $I = q_0/SR$, which gradually diminishes toward zero as q_0 diminishes. The potential energy is dissipated by friction during the process. The relation between q, I, and t is shown in Fig. 116, I.

B. If a constant torque Ψ is applied to the cylinder at rest in its equilibrium position, it will suddenly acquire an angular velocity $I = \Psi/R$, which will decrease toward zero, owing to the return torque exerted by the wire, as the angular displacement increases toward the limiting value $q_0 = S\Psi$. The potential energy will increase during the process toward the limiting value $\frac{1}{2}q_0^2/S$, and an equal quantity of energy will be dissipated by friction. The relations between q, I, and t are shown in Fig. 116, II.

II. Let the elastance of the wire be negligible, or let the wire be removed and let the cylinder be supported on pivots (W/T sensibly zero). A. If a constant torque Ψ is applied to the cylinder, its velocity and kinetic energy will increase from zero toward the limiting values $I = \Psi/R$ and $T = \frac{1}{2}LI^2$. The torque will continue to dissipate energy, the limiting rate being Ψ^2/R.

B. If the torque is suddenly removed, the cylinder will continue to rotate with continually diminishing velocity and energy until the energy is wholly dissipated in heat by friction.

The relations between I and t for the two cases are given in Fig. 117.

III. Let both the elastance of the wire and the inertia of the cylinder be noticeable (W and T both appreciable). A and B. Let $R^2/4L^2$ be equal to or greater than $1/SL$. If the cylinder,

initially displaced though an angle q_0, the potential energy being $\frac{1}{2}q_0^2/S$, is suddenly released, the potential energy will decrease, the velocity and kinetic energy will increase, reach a maximum, and decrease, with q, indefinitely toward zero, all the energy being finally converted into heat.

If the torque is suddenly applied to the cylinder at rest in its equilibrium position, the velocity and kinetic energy increase gradually from zero to a maximum, decrease, and approach zero as the displacement and potential energy approach their limiting values q_0 and $\frac{1}{2}q_0^2/S$. During the process an amount of energy $\frac{1}{2}q_0^2/S$ is dissipated in heat.

The relations between q, I, and t for the first case are shown in Fig. 118.

C. Let $1/SL$ be greater than $R^2/4L^2$. First assume R to be zero. The initial displacement q_0 and potential energy $\frac{1}{2}q_0^2/S$ will decrease to zero as the velocity I and the kinetic energy T increase to maxima when the cylinder is in the equilibrium position. The kinetic energy will carry the cylinder beyond this position until the displacement is $-q_0$, when the kinetic energy and velocity will be zero and the potential energy equal to its initial value. The same phenomena will then recur in inverse order, and so on indefinitely, the time of a complete oscillation being $T_0 = 2\pi(SL)^{\frac{1}{2}}$.

When $R^2/4L^2$ is not zero, the phenomena will be similar except that the motion will be damped, each elongation being less than that immediately preceding, since energy is continually dissipated by friction. The time of an oscillation will be increased, and the time between zero elongation and zero velocity will not be exactly one quarter of a period.

The relations between q, I, and t are shown in Fig. 119.

IV. If an alternating torque $\Psi = \Psi_0 \cos pt$ is applied to the cylinder, it will make simple harmonic vibrations, after the motion has become steady, with the period $T = 2\pi/p$ of the applied e.m.f. (99) may be written

$$q/S = V = \Psi_0 \cos pt - Ld^2q/dt^2 - Rdq/dt$$

If T/T_0 is very great, the velocity $I = dq/dt$, the acceleration d^2q/dt^2, and the second and third terms of the second member, which are the counter torques of inertia and friction, are very small, so that $V = q/S$, the return torque of the twisted wire, is approximately equal to Ψ, the applied e.m.f., and its maximum value V_0 is approximately equal to Ψ_0, while the maximum elongation is approximately equal to $S\Psi_0$.

If T/T_0 is very small, or p/p_0 very great, the counter torque of inertia $-Ld^2q/dt^2$ is great; for the acceleration d^2q/dt^2 is proportional to p and to the amplitude of the velocity dq/dt and is therefore great even if this amplitude is small. Hence the maximum values of q and V, the twist and torque of the wire, are small. The velocity dq/dt is also small.

If $T = T_0$, the torque of inertia is just balanced by the return torque of the wire, or the first member of the above equation and the second term of the second member cancel. Hence the velocity reaches its maximum value, the whole torque $\Psi_0 \cos pt$ being applied to keeping up the velocity or overcoming friction. The amplitude of the velocity, and therewith the maximum elongation of the cylinder and return torque of the wire, increase until the rate of dissipation of energy by friction is equal to the rate at which energy is supplied by the e.m.f. Ψ. If the friction is small, the maximum elongation and return torque may be much greater than $S\Psi_0$ and Ψ_0, which would be produced by a steady torque Ψ_0.

46. The E.M.F. Developed by Rotating a Coil in the Earth's Magnetic Field. One of the most obvious ways of developing a simple harmonic e.m.f. of known period and amplitude is to rotate with constant angular velocity about a vertical axis in the earth's magnetic field a coil of insulated wire wound upon a rigid frame with the planes of its turns parallel and vertical, the ends of the coil being connected to conducting rings revolving on the same axis, the springs bearing on these rings, connecting the terminals of the coil with the rest of the circuit.

Let n denote the number of turns in the coil, S the average area enclosed by a single turn, p the angular velocity, $\theta = pt$ the angle made by the planes of the turns with the magnetic meridian at the time t, and $\mathbf{B} = \mu\mathbf{H}$ the horizontal component of the earth's magnetic induction (sensibly equal in magnitude to $\mathbf{H}$, the horizontal component of the intensity) in the region occupied by the coil.

Then at the time t the coil flux due to the earth's field is

$$N = nS\mathbf{B} \sin \theta$$

and the intrinsic e.m.f. developed in the coil by its rotation is

$$\begin{aligned} \Psi &= -dN/dt = -nS\mathbf{B} \cos \theta \, d\theta/dt \\ &= -pnS\mathbf{B} \cos pt = \Psi_0 \cos pt \end{aligned} \qquad (107)$$

For an accurate determination of a resistance in absolute measure by a method involving essentially the use of such a coil with known constants, see Lord Rayleigh, *Phil. Trans.*, 1882, Part II.

47. A Simple Harmonic E.M.F. Acting on Two Conductors Without Mutual Inductance Connected in Multiple. If an e.m.f.

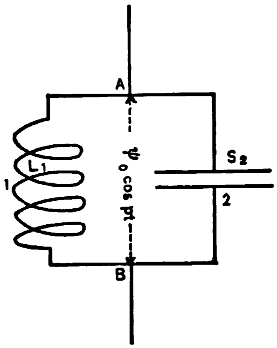


Fig. 121.

$\Psi_0 \cos pt$ is applied to the terminals of the conductors, 1 and 2, it is clear from § 44 that the currents I_1 and I_2 in the conductors

will be simple harmonic with the period $T = 2\pi/p$ of the e.m.f., and that their amplitudes A_1 and A_2 will be in the inverse ratio of the impedances. Thus

$$\frac{A_1}{A_2} = \frac{[R_2^2 + p^2(L_2^2 - 1/S_2 p^2)^2]^{\frac{1}{2}}}{[R_1^2 + p^2(L_1^2 - 1/S_1 p^2)^2]^{\frac{1}{2}}}. \qquad (108)$$

Suppose $L_2 = S_1 = R_1 = R_2 = 0$, approximately (Fig. 121). Then

$$A_1/A_2 = 1/p^2 S_2 L_1 \qquad (109)$$

approximately.

When p is such that $p^2 S_2 L_1 = 1$, or $T = 2\pi(S_2 L_1)^{\frac{1}{2}} = T_0$, the natural period in which the system 12 would execute electrical oscillations if isolated and without resistance, etc., $A_1 = A_2 = A$. The currents in the two branches have in this case the same direction around the circuit 12, and the e.m.f. Ψ cos pt can be removed after the oscillations are started without affecting the phenomena (dissipation and radiation of energy being supposed zero). That I_1 and I_2 are in this case opposite when measured from A to B, or have the same direction around the circuit 12, is clear from (100). For in this case we have

$$\theta_1 = \tan^{-1}(-R_1/pL_1) = \pi - \tan^{-1}R_1/pL_1 = \pi$$

and

$$\theta_2 = \tan^{-1}pR_2S_2 = 0$$

so that

$$I_1 = A\cos(pt - \theta_1 + \tfrac{1}{2}\pi) = A\cos(pt - \tfrac{1}{2}\pi)$$
$$= -A\cos(pt + \tfrac{1}{2}\pi) = -A\cos(pt - \theta_2 + \tfrac{1}{2}\pi) = -I_2$$

Since the current in the external circuit connected at A and B is $I_1 + I_2 = 0$, no power is supplied by the external e.m.f. This also follows from the consideration that I_1 lags 90° behind Ψ and I_2 is in the lead of Ψ by 90°.

This is another example of electrical resonance.

48. Distribution of the Total Discharge Through Two Coils in Multiple. Let the terminals of the two coils, 1 and 2, be joined

at A and B, Fig. 122, and let the e.m.f. from A to B at the time t be denoted by Ψ. The discharge may be that from a condenser, as shown in the figure, or any other form of discharge, and the

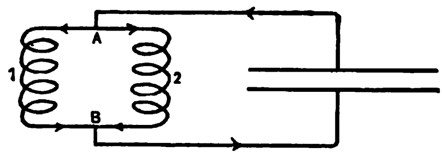


Fig. 122.

circuits may have any inductances and mutual inductance whatever. The capacity in each coil is supposed negligible, so that the current is the same across every section of each conductor, no charge accumulating at any point.

The total charge traversing coil 1 is

$$q_1 = \int I_1 dt = \int \Psi / R_1 \, dt = 1/R_1 \int \Psi \, dt$$

and the total charge traversing circuit 2 is

$$q_2 = \int I_2 dt = 1/R_2 \int \Psi dt$$

Ψ, I_1, and I_2 being considered positive when directed from A to B.

Thus the total discharge $q = q_1 + q_2$ is divided between the two coils in the ratio

$$q_1/q_2 = R_2/R_1 \tag{110}$$

that is, in the inverse ratio of their resistances. Thus the total discharge (considered positive when in one direction, negative when in the other) does not depend upon the inductances, but only upon the resistances.

49. An Electrical System (Transformer) Consisting of Two Circuits, One of Which Contains an Intrinsic E.M.F. Ψ. We shall consider only the case in which the capacities of both circuits are negligible. Let the resistances and inductances of the two cir-

cuits be denoted by R_1, R_2, L_1, L_2, respectively, and their mutual inductance by M; and let the time be reckoned from the instant at which the e.m.f. is applied (to circuit 1), or removed, or the resistance of circuit 1 suddenly altered.

Case I. Ψ = Constant. The total energy of the system is electro-kinetic and equal to

$$W = \tfrac{1}{2}L_1I^2 + MI_1I_2 + \tfrac{1}{2}L_2I_2^2$$

The rate at which energy is dissipated in heat in the two circuits is

$$F = F_1 + F_2 = R_1I_1^2 + R_2I_2^2$$

The rate at which power is supplied to the two circuits is

$$P = \Psi I_1$$

Therefore, by the principle of the conservation of energy,

$$P = dW/dt + F$$

or

$$I_1(L_1dI_1/dt + MdI_2/dt + R_1I_1 - \Psi) \\ + I_2(L_2dI_2/dt + MdI_1/dt + R_2I_2) = 0$$

We proceed to show, quite independently of what precedes, that each of the expressions within parentheses in this equation vanishes separately; that is, that

$$L_1dI_1/dt + MdI_2/dt + R_1I_1 - \Psi = 0 \tag{111}$$

and

$$L_2dI_2/dt + MdI_1/dt + R_2I_2 = 0 \tag{112}$$

equations from which I_1 and I_2 are determined below.

The validity of these equations can be established in several ways. Thus the rate at which electromagnetic energy is generated in circuit 1 plus the rate at which energy is transferred from circuit 2 to circuit 1 is $(\Psi - MdI_2/dt)I_1$, while the rate at which the electromagnetic energy of circuit 1 increases plus the rate at which energy is dissipated by its resistance is $d(\tfrac{1}{2}L_1I_1^2)/dt + R_1I_1^2$. Hence

$$(\Psi - MdI_2/dt)I_1 = d(\tfrac{1}{2}L_1I_1^2)/dt + R_1I_1^2$$

from which (111) results immediately.

(111) also follows immediately from Ohm's law, the impressed e.m.f. in the circuit being $\Psi - L_1dI_1/dt - MdI_2/dt$.

(112) is obtained in exactly the same way as (111), the intrinsic e.m.f. in circuit 2 being put equal to zero.

If we place

$$I_1 - \Psi/R_1 = I_1'$$

(111) and (112) may be written

$$L_1dI_1'/dt + MdI_2/dt + R_1I_1' = 0 \qquad (a)$$

and

$$L_2dI_2/dt + MdI_1'/dt + R_2I_2 = 0 \qquad (b)$$

To solve these equations, put

$$I_1' = Ae^{mt} \text{ and } I_2 = Be^{mt} \qquad (c)$$

where A, B, and m are constants to be determined, and substitute in (a) and (b). On dividing the resulting equations by e^{mt}, we obtain, as the conditions which the constants must satisfy in order that (c) may give the solution,

$$(L_1m + R_1)A + MmB = 0 \qquad (d)$$

$$MmA + (L_2m + R_2)B = 0 \qquad (e)$$

Eliminating A and B, we obtain, to determine m, the equation

$$(L_1L_2 - M^2)m^2 + (R_2L_1 + R_1L_2)m + R_1R_2 = 0 \qquad (f)$$

whose roots are

$$m_1 = \frac{-(R_2L_1 + R_1L_2) + [(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2]^{\frac{1}{2}}}{2(L_1L_2 - M^2)} \qquad (g)$$

and

$$m_2 = \frac{-(R_2L_1 + R_1L_2) - [(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2]^{\frac{1}{2}}}{2(L_1L_2 - M^2)} \qquad (h)$$

Since $(R_2L_1 - R_1L_2)^2 = (R_2L_1 + R_1L_2)^2 - 4R_1R_2L_1L_2$, and since $L_1L_2 - M^2 \geqq 0$, the quantity under the radical is positive and numerically less than, or equal to, the first term of the numerator, and the denominators are positive; hence *both* m_1 and m_2 are *real* and *negative*.

Making use of (g) and (h), we can now obtain from (d) and (e) the values of A/B satisfying (a) and (b). Thus, putting $m = m_1$ in (d) and (e), and denoting the resulting value of A/B by A_1/B_1, we obtain

$$A_1/B_1 = -Mm_1/(L_1m_1 + R_1) = -(L_2m_1 + R_2)/Mm_1 \quad (i)$$

and similarly, putting $m = m_2$,

$$A_2/B_2 = -Mm_2/(L_1m_2 + R_1) = -(L_2m_2 + R_2)/Mm_2 \quad (j)$$

Thus the general solutions of (111), (112), (a), and (b) are

$$I_1' = I_1 - \Psi/R_1 = A_1e^{m_1t} + A_2e^{m_2t} \quad (k)$$

and

$$\begin{aligned} I_2 &= B_1e^{m_1t} + B_2e^{m_2t} \\ &= -[A_1(L_1m_1 + R_1)/Mm_1 \cdot e^{m_1t} + A_2(L_1m_2 + R_1)/Mm_2 \cdot e^{m_2t}] \end{aligned} \quad (l)$$

where A_1 and A_2 are to be determined from the initial conditions.

A. The circuit containing Ψ is suddenly closed. The initial conditions are $I_1 = I_2 = 0$ when $t = 0$. Hence, from (i), (j), (k), and (l), we obtain

$$\begin{aligned} I_1 = \frac{\Psi}{R_1}\Bigg(1 - \tfrac{1}{2}\left\{\frac{(R_2L_1 - R_1L_2)}{[(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2]^{\frac{1}{2}}} + 1\right\}e^{m_1t} \\ + \tfrac{1}{2}\left\{\frac{R_2L_1 - R_1L_2}{[(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2]^{\frac{1}{2}}} - 1\right\}e^{m_2t}\Bigg) \end{aligned} \quad (113)$$

and

$$I_2 = -\Psi M/[(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2]^{\frac{1}{2}} \cdot (e^{m_1t} - e^{m_2t}) \quad (114)$$

The relations between I_1 and I_2 and the time after closing the circuit 1 are shown approximately for the general case in Fig.

123. I_1 approaches asymptotically a final value Ψ/R_1, and I_2 has a maximum at the time given by $dI_2/dt = 0$.

The total charge traversing circuit 2, obtained either by integrating (114) from the time $t = 0$ to the time $t =$ infinity, or from (12) directly, is

$$\begin{aligned} q = \int I_2 dt = -(N_2 - N_1)/R_2 &= -M(\Psi/R_1 - 0)/R_2 \\ = -M\Psi/R_1R_2 &= -MI/R_2 \end{aligned} \qquad (115)$$

if I denotes the final current Ψ/R_1 in circuit 1.

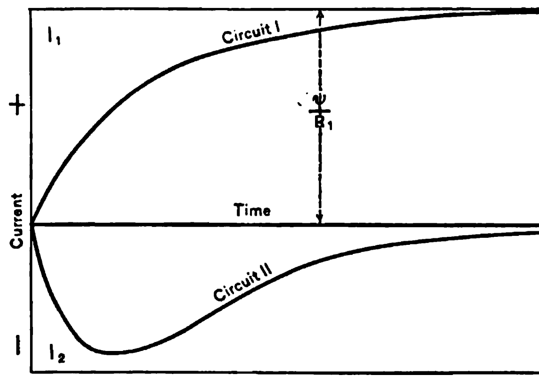


Fig. 123.

B. The electromotive force Ψ is suddenly cut out of circuit 1 (without opening the circuit). The values of the currents, in terms of I_1 and I_2, given in (113) and (114), are

$$I_1'' = \Psi/R_1 - I_1 = -I_1' \qquad (116)$$

and

$$I_2'' = -I_2 \qquad (117)$$

t now denoting the time reckoned from the instant of cutting out the electromotive force.

50. Case II. The Electromotive Force is a Simple Harmonic Function of the Time, Ψ_0 cos pt. In this case we have

$$L_1dI_1/dt + MdI_2/dt + R_1I_1 = \Psi_0 \cos pt \qquad (118)$$

and

$$L_2dI_2/dt + MdI_1/dt + R_2I_2 = 0 \qquad (119)$$

The general solution of these equations is that given in (k) (for I_1') and (l) added to a particular solution now to be obtained.

Since the applied e.m.f. is harmonic, we may assume that the currents to which it gives rise will also be harmonic in the same period. Hence we put

$$I_1 = A \cos (pt - \theta_1) \tag{120}$$

and

$$I_2 = B \cos (pt - \theta_2) \tag{121}$$

where A, B, θ_1 and θ_2 are constants to be determined. To determine these constants, substitute (120) and (121) in (118) and (119), and equate to zero the coefficients of $\cos pt$ and $\sin pt$ separately, since the resulting equations are true for all values of t. Thus, putting

$$L' = L_1 - p^2M^2L_2/(R_2^2 + p^2L_2^2) \tag{122}$$

and

$$R' = R_1 + p^2M^2R_2/(R_2^2 + p^2L_2^2) \tag{123}$$

we find, as conditions that (120) and (121) may be solutions of (118) and (119),

$$A = \Psi_0/(R'^2 + p^2L'^2)^{\frac{1}{2}} \tag{124}$$

$$B = MpA/(R_2^2 + p^2L_2^2)^{\frac{1}{2}} \tag{125}$$

$$\theta_1 = \tan^{-1} pL'/R' \tag{126}$$

$$\theta_2 = \theta_1 + \tan^{-1} (- R_2/pL_2) \tag{127}$$

which, inserted in (120) and (121), give the particular solution sought.

The terms given by (k) and (l) become negligible in a very short time, hence after this time (120) and (121) are the complete solution of (118) and (119).

On comparing (120) and (124) with (103), we see that the current in circuit 1 is the same as it would be if circuit 2 were not present and the resistance of circuit 1 were increased from R_1 to R', and its inductance diminished from L_1 to L'.

When p^2L_2 is great in comparison with R_2, (122) and (123) become approximately

$$L' = L_1 - M^2/L_2 \quad \text{and} \quad R' = R_1 + R_2M^2/L_2^2$$

If in addition the coils are wound together so that the *magnetic leakage* is zero, or $L_1L_2 = M^2$, approximately, L' becomes approximately zero. In this case, $\theta_1 = 0$, the current in circuit 1 being in phase with the e.m.f., and $\theta_2 = \pi$, so that the current in circuit 2 lags behind the current (and e.m.f.) in circuit 1 by half a period.

If either circuit contains in series a condenser of capacity S, the solution is obtained, as follows from § 44, by writing in place of L_1 or L_2, $L_1 - 1/Sp^2$ or $L_2 - 1/Sp^2$, in the expressions for A, B, θ_1, and θ_2.

If the magnetic field contains iron, or other substance in which hysteresis occurs, a simple harmonic applied e.m.f. will not develop a simple harmonic current. For the current is proportional to the magnetic intensity H, by the first law of circuitation, and the induced part of the total e.m.f. is proportional to the rate of change dB/dt of the magnetic induction B. But if dB/dt is a simple harmonic function of the time, then, if there is hysteresis, H cannot be, as is evident from Fig. 114. This of course applies to single circuits as well.

51. The Ratio of Transformation of Two Coils for Which $L_1L_2 = M^2$ and $R_1 = R_2 = 0$, approximately. Let two coils 1 and 2 be so wound that all the magnetic flux Φ threads every turn of both coils (*magnetic leakage* = 0). Let coil 1 contain n_1 turns and coil 2 n_2 turns; then the flux through coil 1 will be $N_1 = n_1\Phi$ and the flux through coil 2 will be $N_2 = n_2\Phi$.

The e.m.f. applied between the terminals of coil 1 will be

$$\Psi_1 = R_1I_1 + dN_1/dt$$

and the e.m.f. between the terminals of coil 2 will be

$$\Psi_2 = -dN_2/dt - R_2I_2$$

the same direction being chosen as positive around both circuits.

Hence if R_1I_1 and R_2I_2 are negligible in comparison with the electromotive forces of induction, we have, in magnitude,

$$\Psi_1/\Psi_2 = n_1 d\Phi/dt/n_2 d\Phi/dt = n_1/n_2 \qquad (128)$$

which is called the *ratio of transformation of e.m.f.s* of the two coils.

R_1 and R_2 above denote only the resistance of the *coils*, or those parts of circuits 1 and 2 threaded by the tubes of magnetic induction common to both circuits.

The power supplied to the system by the e.m.f. Ψ_1 is $\Psi_1 I_1$, and the power supplied to that part of circuit 2 outside the coil is $\Psi_2 I_2$. When no energy is dissipated in the coils we have

$$\Psi_1 I_1 = \Psi_2 I_2$$

whence

$$I_1/I_2 = \Psi_2/\Psi_1 = n_2/n_1 \qquad (129)$$

which is the *ratio of transformation of currents* for the case considered.

52. Electromagnetic Repulsion Between Two Circuits. When an alternating current is induced in a circuit (2) owing to the circulation of an alternating current in a neighboring circuit (1) connected with an alternating current generator, a force is developed between the two circuits, positive or repulsive when the two currents have opposite directions, and negative or attractive when the two currents have the same direction. We proceed to show that the *average* force is one of repulsion.

For the sake of simplicity let the two circuits be circular with their axes coincident, and let circuit (2) consist of a single turn of wire. Let the magnetic flux through circuit (2) be a harmonic function of the time,

$$N = A \cos nt$$

Then the outward radial component of the magnetic induction at all points of circuit (2) will be

$$B_r = a \cos nt$$

where a is a constant

The e.m.f. induced in circuit (2) will be

$$\Psi = -dN/dt = nA \sin nt$$

and the current in circuit (2),

$$I = C \sin (nt - \theta)$$

where C and θ are positive constants ($\theta < 90°$).

The force upon circuit (2) in the direction 1 2 is therefore

$$F = -\mathrm{V}IB_r \times \text{length of circuit (2)}$$
$$= -\cos nt \cdot \sin (nt - \theta) \times \text{positive constant}$$

When $\cos nt$ and $\sin (nt - \theta)$ have the same sign, or opposite signs, the force upon coil (2) is thus an attraction toward (1), or a repulsion from (1), respectively. On platting $\cos nt$, $\sin(nt - \theta)$, and their product, which is proportional to $-F$, as functions of the time, it will be seen that, owing to the *lag* of the current, the average force is a *repulsion* between the circuits.

The same result follows from integrating Fdt throughout a complete period, the average force being

$$\frac{1}{T}\int_0^T Fdt = -\frac{1}{T}\int_0^T \cos nt \ \sin (nt - \theta) \cdot dt \times \text{positive constant}$$

Analogous theory applies to a great variety of cases, one of the simplest being that of a light coil, or disc, of copper, aluminium, or other good conductor placed horizontally over an electromagnet with coils horizontal. When the magnet is powerfully excited by an alternating current the disc or coil is thrown violently up into the air. Another interesting case is that of a horizontal copper disc mounted on a pivot and placed eccentrically over the electromagnet, with a second disc, held likewise horizontal, near by. Owing to the asymmetry thus introduced, the pivoted disc rotates continuously while the magnet is powerfully excited.

53. The Comparison of a Mutual Inductance and a Resistance One (1) of the two coils, of mutual inductance M, is connected

in circuit with a constant battery B through a key K, while the other (2) is connected in circuit with a ballistic galvanometer, the total resistance of circuit (2) being R.

When a current I is suddenly started or stopped in coil (1) by closing or opening the key K, a charge $q = MI/R$ circulates through coil (2) and produces an angular throw θ of the needle such that

$$MI/R = \mathbf{H}T/G\pi \cdot \sin \tfrac{1}{2}\theta$$

or

$$M/R = \mathbf{H}/GI \cdot T/\pi \cdot \sin \tfrac{1}{2}\theta \tag{130}$$

Circuit (2) is now cut at some point and the resulting terminals connected to two points on circuit (1) with a resistance r, very small in comparison with the total resistance of circuit (1), between them. This will not sensibly affect the current in circuit (1), but will cause a steady current

$$r/(r + R) \cdot I = \mathbf{H}/G \cdot F(\theta') \text{ (see § 33, XII.)} \tag{131}$$

to traverse the galvanometer, producing the steady deflection θ'. Eliminating $\mathbf{H}/GI$ from (130) by (131), we obtain, finally,

$$M/R = r/(r + R) \cdot T/\pi \cdot \sin \tfrac{1}{2}\theta/F(\theta') \tag{132}$$

θ, $F(\theta')$, T, and the ratio of r to $r + R$ being observed, M/R is given in absolute measure by (132).

Absolute Determination of a Resistance. From the dimensions of the coils, if circular, toroidal, or rectangular, M can be calculated; hence the method affords an absolute determination of a resistance. See Glazebrook, *Phil. Trans.*, 1883, for an important investigation in which this method was adopted.

The Comparison of Two Mutual Inductances. By sending the same current in succession through the primaries of two coils and connecting the secondaries in succession through the same galvanometer, the total resistances of the secondary circuits in the two cases being R_1 and R_2, the two mutual inductances M_1 and M_2, and the two deflections θ_1 and θ_2, the ratio of the two mutual inductances can be obtained from the relation

$$M_1/M_2 = R_1 \sin \tfrac{1}{2}\theta_1/R_2 \sin \tfrac{1}{2}\theta_2 \tag{133}$$

By putting an adjustable resistance in one circuit, or in each circuit, and changing the resistance until $\theta_1 = \theta_2$, we have

$$M_1/M_2 = R_1/R_2 \tag{134}$$

By connecting the two secondaries in series with the galvanometer permanently, and passing the same current in succession through the two primaries, we have

$$M_1/M_2 = \sin \tfrac{1}{2}\theta_1 / \sin \tfrac{1}{2}\theta_2 \tag{135}$$

The Comparison of Resistances. The method can also be employed for the comparison of two resistances in the secondary circuits. In this case

$$R_1/R_2 = \sin \tfrac{1}{2}\theta_2 / \sin \tfrac{1}{2}\theta_1 \tag{136}$$

54. The Measurement of Magnetic Induction by the Ballistic Method. A small coil consisting of n turns of fine insulated wire uniformly wound in parallel planes, the mean area enclosed by a single turn being S, is connected by twisted wires to the terminals of a ballistic galvanometer. The coil is introduced into the magnetic field at the place where the induction is to be determined with the planes of its turns perpendicular to the induction, and the galvanometer needle is brought to rest. Then the coil is suddenly jerked out of the magnetic field, and the resulting deflection θ of the galvanometer needle is read. Let B denote the mean induction perpendicular to the planes of the coil's turns in the part of the field in which the coil was placed, and let R denote the resistance of the circuit. Then, by (12), and (49) XII., the charge traversing the circuit and producing the deflection θ is

$$q = nSB/R = \mathbf{H}T/\pi G \cdot \sin \tfrac{1}{2}\theta$$

from which

$$B = R\mathbf{H}T/\pi nSG \cdot \sin \tfrac{1}{2}\theta \tag{137}$$

If μ (sensibly equal to unity in nearly all cases) is known, the magnetic intensity can be found by dividing B by μ.

55. Maxwell's Method of Comparing Two Mutual Inductances. In the practise of this method (Fig. 124) two of the coils (the primaries), one from each pair, are connected in series with the battery through a key *K*, and the terminals of each of the other coils (the secondaries) are connected to the terminals of a galvanometer *G* through an adjustable resistance, connections being so arranged that the discharges of the two secondaries, on

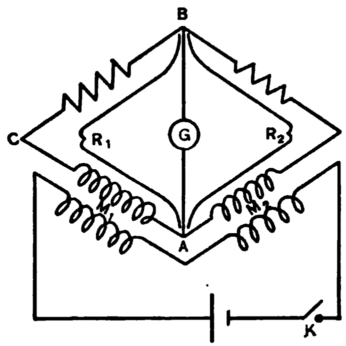


Fig. 124.

opening or on closing *K*, when tested separately, traverse the galvanometer in opposite directions. Then the resistances in the secondaries are adjusted until the galvanometer needle remains undeflected, *i. e.*, until the *total* discharge through the galvanometer reckoned in one direction is zero, when *K* is opened or closed. Then, if R_1 and R_2 denote the adjusted resistances from *A* to *B* through *C* and from *A* to *B* through *D*, respectively,

$$M_1/M_2 = R_1/R_2 \tag{138}$$

To prove this, let I denote the current through the battery and primaries, I_1, I_2, and $I_3 = I_1 - I_2$ the currents from *A* to *B* through *C*, from *A* to *B* through *D*, and from *A* to *B* through the galvanometer, at the time t; and let L_1, L_2, and L_3 denote the inductances of *ACB*, *ADB*, and *AGB*, and R_3 the resistance of *AGB*. Then the impressed e.m.f. from *A* to *B* at the time t is

$$M_1dI/dt - L_1dI_1/dt - R_1I_1 = L_3d(I_1 - I_2)/dt + R_3(I_1 - I_2)$$

and the impressed e.m.f. from B to A at the time t is

$$M_2 dI/dt - L_2 dI_2/dt - R_2 I_2 = L_3 d(I_2 - I_1)/dt + R_3(I_2 - I_1)$$

Multiplying each equation by dt and integrating from the time of closing the circuit ($t = 0$) to the time at which the battery current becomes steady ($t = t'$), remembering that the initial and final values of I_1 and I_2 are zero and that $\int_0^{t'} (I_1 - I_2)dt$, the total discharge through the galvanometer, is zero, and denoting the steady value of the battery current by I_0, we have

$$M_1 I_0 - R_1 \int_0^{t'} I_1 dt = 0$$

$$M_2 I_0 - R_2 \int_0 I_2 dt = 0$$

from which (138) immediately follows, since

$$\int_0^{t'} I_1 dt = \int_0^{t'} I_2 dt$$

The Comparison of Resistances. We can vary the method slightly for the comparison of two resistances. Suppose the balance given by (138) effected. Introduce an unknown resistance X into the branch ADB, and balance by adding a known resistance R_1' to R_1. Then

$$M_1/M_2 = (R_1 + R_1')/(R_2 + X)$$

Combining this equation with (138) we find

$$X/R_1' = R_2/R_1 = M_2/M_1 \qquad (139)$$

If X and R_1' are both known, (139) gives M_2/M_1 without a knowledge of R_2/R_1.

56. Maxwell's Method of Comparing an Inductance and a Capacity. The given coil, with inductance L and resistance R, is connected up in a Wheatstone's bridge with three non-inductive

resistances P, Q, T, Fig. 125, and a resistance balance for steady currents is obtained in the usual way by adjusting the resistances until the galvanometer needle remains undeflected when K_2 is

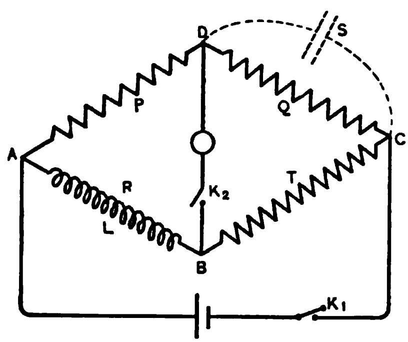


Fig. 125.

closed after K_1. Then, if V_{ab} denotes the fall of potential from A to B, etc., and I the current through ABC,

$$V_{ab}/V_{bc} = RI/TI = R/T = V_{ad}/V_{dc} = P/Q$$

During the variable state of the currents just after closing or opening K_1, however, K_2 being open,

$$V_{ab}/V_{bc} = (RI + LdI/dt)/TI \gtrless R/T = V_{ad}/V_{dc}$$

Hence, if K_1 is opened or closed while K_2 is closed, the galvanometer needle will be deflected.

If, however, a condenser is introduced as a shunt to DC, as shown by the dotted lines, a part of the current through P will be shunted into this condenser during the rise of the current after the closing of K_1, thus reducing the current through Q and increasing the ratio V_{ad}/V_{dc}; and during the decrease of the current after the opening of K_1 the discharge of the condenser will increase the current through Q and decrease the current through P, thus diminishing the ratio V_{ad}/V_{dc}. Hence, since the law of the increase or decrease of the induced current in an inductive resistance with the time, and the law of the increase or decrease

of the charging current of a condenser with the time, are identical, §§ 41–42, an exact balance for both transient and steady currents can be obtained by using, with given inductance and resistances, a condenser of a particular capacity. By readjusting the non-inductive resistances, however, a balance for both steady and transient currents may be obtained for any inductance and capacity. Thus, if R and P are fixed, the ratio of the effect of the inductance to that of the capacity is decreased by increasing Q and T and keeping always $Q/T = P/R$. When this double balance has been attained, then, as shown below,

$$L/S = RQ = PT \tag{140}$$

Anderson's Modification. To avoid the necessity of this tedious process of readjustment and trial, an additional non-inductive resistance W may be inserted between D and the condenser and galvanometer, Fig. 126. This will not affect the balance for

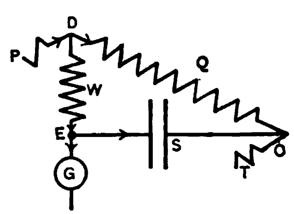


Fig. 126.

steady currents, but will enable the effect of the condenser on the balance for variable currents to be altered. After the balance for steady currents has been attained, the resistance W is altered until the balance is good for transient currents also. Then

$$L/S = W(R + T) + PT \tag{141}$$

W thus increases the effect of the condenser. If the condenser has too great an effect when $W = 0$, PT (or QR) must be decreased in the balance for steady currents.

To establish (141), let x, x, $y + z$, y, and z denote the currents in the branches R, T, P, Q, and W, respectively, at the time t after closing or opening K_1, the double balance having been attained.

Then, since the fall of potential along the path ABE is equal to that along the path ADE, we have

$$Rx + Ldx/dt = P(y+z) + Wz \qquad (a)$$

In the same way the fall of potential from D to C through Q is equal to that from D to C through W and the condenser; that is

$$Qy = Wz + \left(q_0 + \int_0^t zdt\right)/S \qquad (b)$$

where q_0 denotes the initial charge of the condenser.

Likewise, the fall of potential from E to C through the condenser is equal to that from E to C through the galvanometer and T; or

$$\left(q_0 + \int_0^t zdt\right)/S = Tx \qquad (c)$$

Eliminating x and y from (a) by (b) and (c), differentiating with respect to t, and equating to zero separately the coefficients of z and dz/dt (since the equation holds for all values of z and dz/dt, including zero), we obtain the conditions of a double balance:

$$R/T = PQ \qquad (d)$$

the condition for a balance for steady currents; and (141), *viz.*,

$$L/S = (R+T)W + PT$$

which reduces to (140) when $W = 0$.

Russell's Modification. If a given inductance is to be compared with an adjustable standard capacity, or if a capacity is to be compared with a standard inductance which can be adjusted to different values without an alteration of resistance, the balance for steady currents is first effected, then the galvanometer circuit is closed and the standard capacity or inductance altered until the galvanometer needle remains undeflected when the battery key is opened or closed.

57. Maxwell's Method of Comparing Inductances. The two coils whose inductances L_1 and L_2 are to be compared are joined up with four non-inductive adjustable resistances and a battery and galvanometer as shown in Fig. 127. (The connections of

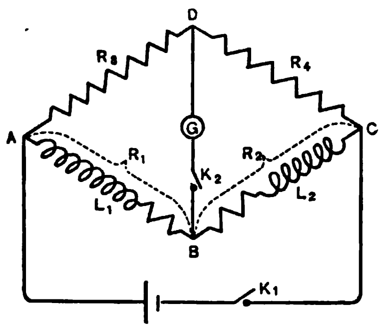


Fig. 127.

the battery and galvanometer may be interchanged.) R_1, R_2, etc., denote the total resistances in the branches AB, BC, etc. The resistances are varied until the galvanometer shows no deflection for steady currents (K_2 closed after K_1) or for transient or variable currents (K_1 closed after K_2 or opened before K_2). Then

$$L_1/L_2 = R_1/R_2 = R_3/R_4 \tag{142}$$

For, when such a double balance has been effected, the voltage from A to B is equal to the voltage from A to D at any time t; that is,

$$R_1I_1 + L_1dI_1/dt = R_3I_3$$

where I_1 and I_3 denote the currents through ABC and ADC respectively. In like manner the voltage from B to C is equal to the voltage from D to C, that is

$$R_2I_1 + L_2dI_1/dt = R_4I_3$$

Combining these two equations and dividing the resulting equation by I_3, we obtain

$$R_1R_4I_1 + L_1R_4dI_1/dt = R_2R_3I_1 + L_2R_3dI_1/dt$$

Since this equation is true for all values of the current I_1 and its rate of change, including zero, we have

$$R_1R_4 = R_2R_3 \tag{a}$$

which is the condition for a balance with steady currents, and

$$L_1R_4 = L_2R_3 \tag{b}$$

which is the additional condition for a balance for variable currents. From these equations (142) follows immediately.

Standard coils are constructed whose inductances can be varied within wide limits without an alteration of resistance. If one of the coils to be compared is a standard of this kind, a balance for steady currents is first obtained in the ordinary way; then, without altering the resistance in any part of the network, a balance for transient currents is made by altering the inductance of the standard. See § 18.

58. Cary Foster's Methods of Comparing a Mutual Inductance and a Capacity. Let connections be made as in Fig. 128, S and

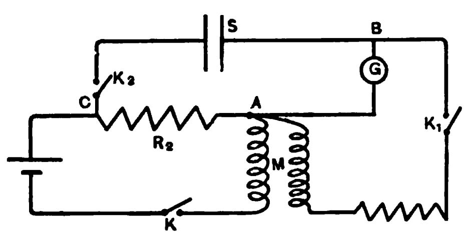


Fig. 128.

M denoting the capacity and mutual inductance, R_1 the total resistance from A through K_1 to B, and R_2 the resistance between the points A and C, both resistances being, at least in part, adjustable. If K_1 is open and K_2 closed, then when K is opened or closed a charge SR_2I will traverse the galvanometer, I denoting the steady value of the current in the battery circuit. If K_2 is open and K_1 closed, a charge $MI/(R_1 + g)$, where g denotes the galvanometer resistance, will traverse the galvanometer.

By reading the two galvanometer deflections the ratio of M to S may therefore be obtained.

In Cary Foster's null method connections are so made that on closing K or on opening K the capacity and inductance discharges are in opposite directions through the galvanometer. Then K_1 and K_2 are both closed, and the resistances R_1 and R_2 are adjusted until on opening or closing K there is no deflection of the galvanometer needle, showing that the total discharge through the galvanometer is zero. Then

$$MI/R_1 = R_2IS$$

g not entering the expression. Hence

$$M/S = R_1R_2 \tag{143}$$

The demonstration is left to the reader, who should refer to § 55.

59. Brillouin's Modification of Maxwell's Method of Comparing the Mutual Inductance, M, of Two Coils with the Self-Inductance, L, of One of Them. The coil with inductance L and resistance R

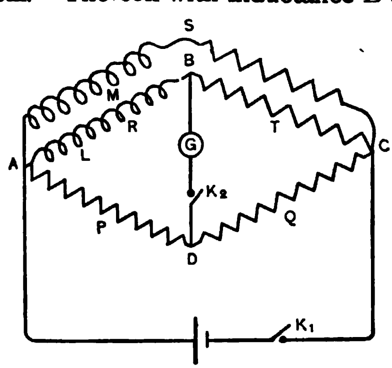


Fig. 129.

is connected up with three non-inductive resistances P, Q, T in a Wheatstone's bridge, Fig. 129, and the other coil is connected through an adjustable resistance to the points A and C. Balance is first obtained for steady currents in the usual way, then the

resistance S of the branch ASC is adjusted until the galvanometer shows no deflection when the galvanometer circuit is closed and the battery key is opened or closed. In order that the mutual inductance of the two coils may thus neutralise the effect of the self-inductance of one of them, the two coils must be so connected that their currents flow always in opposite directions around the tubes of induction which thread them. Then

$$L/M = (R + T)/S \tag{144}$$

For the increase of voltage from A to B due to the self-inductance is $L\,dx/dt$, and the decrease in the voltage due to the mutual inductance is $M\,dz/dt$, where x and z denote the currents through ABC and ASC respectively. Hence, when the balance for transient currents is attained,

$$L\,dx/dt = M\,dz/dt$$

Integrating through the time t in which the currents rise from zero to their steady values x_0 and z_0, or in which they drop from their steady values to zero, we have

$$L\int_0^t dx/dt\,dt = Lx_0 = M\int_0^t dz/dt\,dt = Mz_0$$

Hence $$L/M = z_0/x_0 = (R + T)/S$$

60. The Comparison of a Capacity with a Resistance. A capacity may be compared with a resistance by a method exactly analogous to that of § 53, as shown in § 36, XII. The following null method, due to Maxwell, is, however, much to be preferred.

One branch of the Wheatstone's bridge, as the branch 23, Fig. 76 is cut and a condenser AB inserted, the plate B being connected by a wire of negligible resistance to the point 3, and the plate A being connected in the same way with a rapidly and uniformly moving commutator, which puts it alternately into electrical contact with the point 3 and the point 2. When A is connected to 3 the condenser is short-circuited and the galvanom-

eter is traversed by a current in the direction 24; while during a part of the time the plate A is connected with 2 the galvanometer is traversed by a current in the opposite direction 42, the plate A being charged through the galvanometer and the branch 12. If the period of the galvanometer needle (or coil) is great in comparison with the period of the commutator, a steady deflection will, in general, result. By suitably adjusting the resistances a, b, and d, however, the average current through the galvanometer, reckoned as positive in one direction and negative in the other, may be made zero, when the galvanometer will show no deflection whether the battery is connected to the bridge or not. This is the only adjustment to be made in the practise of the method.

When this balance has been effected, the average voltage from 1 to 2 is equal to the average voltage from 1 to 4, *i. e.*, to

$$b\Psi / (d + b)$$

if Ψ denotes the e.m.f. of the battery, and if the battery resistance is negligible (a condition easy to attain) in comparison with that of the bridge (from point 1 to point 3). Hence the average value of the current in the branch 12 is

$$I = b\Psi / a(b + d)$$

and this is equal to the average charging current of the condenser, since the average current through the galvanometer is zero.

The voltage of the condenser when fully charged is the voltage between the points 1 and 2 when the current in every part of the bridge has reached its steady state. This voltage is readily seen to be

$$\Psi \frac{bg + d(a + b + g)}{b(a + g) + d(a + b + g)}$$

Hence, if n denotes the frequency of the commutator, or the number of times the condenser is charged per second, and S the

capacity of the condenser, the average value of the charging current of the condenser is

$$I = nS\Psi \frac{bg + d(a + b + g)}{b(a + g) + d(a + b + g)}$$

Equating the two expressions for I and solving for S, we obtain

$$S = \frac{b[b(a + g) + d(a + b + g)]^*}{na(b + d)[bg + d(a + b + g)]} \tag{145}$$

When the condenser has a guard ring, this method cannot be used without an inconvenient modification.† A much simpler method, however, equal in accuracy, can be used in all cases. A galvanometer with two independent coils acting on the same needle (differential galvanometer) has the intermittent condenser current sent through one coil and a steady current from the battery in the opposite direction through the other. The deflection is reduced to zero by suitably adjusting the resistance connected with this second coil. This method, due to Klemencic, has been used with great precision by Himstedt ‡ and by H. Abraham § in the determination of a (XIV., § 4).

Comparison of Capacities. These methods also serve admirably for the comparison of capacities, only the ratios of the resistances being necessarily known.

61. The Comparison of an Inductance with a Resistance. The coil AB, with inductance L, is connected up with three non-inductive resistances P, Q, and T, a galvanometer, and a constant battery, as shown in Fig. 125 (the dotted lines being annulled), and a balance for steady currents is effected in the usual way. With the key K_2 closed, K_1 is then either opened or closed, when the flux through the coil will change from LI to 0 or from 0 to

* H. Abraham, *Ann. de Chim. et de Phys.*, Vol. 27, 1892.

† For an important investigation in which the method was modified and used with a guard ring condenser, see Thomson and Searle, *Phil. Trans.*, A, 1890.

‡ *Wied. Ann.*, Vol. 35, 1888.

§ Loc. cit.

LI, I denoting the steady value of the current in AB. Owing to the change of flux and the e.m.f. developed thereby in the coil AB, a charge q proportional to LI will traverse the galvanometer, producing an angular throw θ, such that

$$q = KLI = \frac{\mathbf{H}}{G}\frac{T}{\pi}\sin\tfrac{1}{2}\theta$$

where K is a constant depending on the resistances.

The balance for steady currents is then disturbed by increasing or decreasing the resistance R of the branch AB by a very small quantity ΔR. If I' denotes the new value of the steady current in AB, an e.m.f. $I'\Delta R$ will thus exist in the branch during the steady state, and a steady deflection θ' of the galvanometer needle will result, such that

$$KI'\Delta R = \frac{\mathbf{H}}{G}F(\theta')$$

K being the same constant occurring in the previous equation.

Eliminating K from the two equations and solving for L, we have

$$L = \Delta R \cdot \frac{I'}{I} \cdot \frac{T}{\pi}\frac{\sin\tfrac{1}{2}\theta}{F(\theta')} \qquad (146)$$

The ratio I'/I of the final and initial values of the steady current in AB can be calculated from the resistances in the bridge. It is obvious that ordinarily the ratio will be sensibly equal to

$$\frac{I'}{I} = \frac{R+T}{R+\Delta R+T}$$

which is very nearly unity.

The method was originated by Maxwell, and was first used with precision in an important investigation by Lord Rayleigh (*Phil. Trans.*, Part II., 1882).

CHAPTER XIV.

UNITS AND DIMENSIONS.

1. The Electrostatic Systems of Units. (1) *The rational electrostatic system.* The rational electrostatic unit charge, fully discussed in Chapter I., is defined by the equation

$$q = (4\pi c L^2 F)^{\frac{1}{2}} \qquad (1)$$

where L and F are expressed in c.g.s. units and c is expressed in terms of the permittivity of free æther (c_0) as unit permittivity.

From this fundamental definition and the further definition that all the equations hitherto developed (except those specified as belonging to irrational systems) hold good for all rational systems of units (electromagnetic as well as electrostatic), the definitions of the rational electrostatic units of all the other electrical quantities follow. Thus the definition of the *RES* unit current is given by the equation $I = q/t$, I being expressed in *RES* unit current when q is expressed in the *RES* unit charge and t in the c.g.s. unit time; similarly, the definition of the *RES* unit magnetic intensity then follows from the relation $\Omega =$ m.m.f. $= H \times 2\pi d = I$ (§ 14 or 15, XII.); the definition of the *RES* unit magnetic pole strength then follows from the equation $m = F/H$; and then the definition of the *RES* unit magnetic inductivity from the equation $\mu = m^2/4\pi L^2 F$; etc.

(2) *The common or irrational electrostatic system.* If, however, while the units of permittivity, force, and length remain unchanged, A, equation (1′), I., is put equal to unity instead of 4π, we obtain, as the definition of the (irrational or common) *electrostatic unit charge*, the equation

$$q = (c L^2 F)^{\frac{1}{2}} \qquad (2)$$

On comparing this equation with (1) we see that the electrostatic unit just defined is equal to $(4\pi)^{\frac{1}{2}}$ times the rational unit.

The system of electrical units built up from this unit charge as fundamental unit in a manner exactly similar to that in which the rational system is built up from the *RES* unit charge, with the exception of a few units, such as intensity of electrisation or magnetisation, which, as stated in appropriate places in the text, are differently defined, is called the *electrostatic* (*ES*) system of units.

2. The Electromagnetic Systems of Units. (1) *The rational electromagnetic system.* The rational electromagnetic unit magnetic pole strength is defined by the equation

$$m = (4\pi\mu L^2 F)^{\frac{1}{2}} \tag{3}$$

where L and F are expressed in c.g.s. units and μ is expressed in terms of the inductivity of free æther (μ_0) as unit inductivity.

From this fundamental definition and the general equations already developed for systems defined as rational, the definitions of the rational electromagnetic units of all the other electrical quantities follow. Thus the *REM* unit magnetic intensity is defined by the equation $H = F/m$, H being expressed in the *REM* unit magnetic intensity by definition, when m is expressed in the *REM* unit pole strength and F in c.g.s. units; the *REM* unit current is defined by the relation $I = F/BL$ or $I = 2\pi d \times H$; the *REM* unit electric charge from the relation $q = It$; the *REM* unit electric permittivity from the equation $c = q^2/4\pi L^2 F$; etc.

(2) *The common* (*or irrational*) *system.* If, without changing the units of inductivity, force, or length, we put A, (1), XI., equal to unity instead of 4π, we obtain, as the definition of the *electromagnetic* (*EM*) *unit pole strength*, the relation

$$m = (\mu L^2 F)^{\frac{1}{2}} \tag{4}$$

On comparing this equation with (3) we see that the electromagnetic unit just defined is equal to $(4\pi)^{\frac{1}{2}}$ times the electromagnetic rational unit pole strength.

The system of electrical units built up from this unit pole strength as fundamental unit in a manner exactly similar to that in which the rational system is built up from the *REM* unit pole strength, with the exceptions referred to in the closing paragraph of § 1, is called the *electromagnetic* system of units.

3. Relations Between the Units of Different Systems. Every equation developed in the preceding chapters holds good, as already stated, by definition, when every electrical quantity occurring therein is expressed in its rational electrostatic unit, or when every electrical quantity is expressed in its rational electromagnetic unit, all other quantities being expressed in c.g.s. units. Every one of these equations that contains the definition of a unit, moreover, except those defining unit charge, unit pole strength, and the other units referred to in the closing paragraph of § 1 and mentioned in the appropriate places in the text, is valid also when expressed in irrational units throughout, either all electrostatic or all electromagnetic. Thus, on any system of units, electric intensity is defined as the force per unit charge, magnetic intensity as the force per unit pole strength, electric displacement as permittivity × intensity, capacity as charge per unit voltage, etc. The following definitional equations, in which plain letters denote quantities expressed in rational units and primed letters quantities expressed in irrational units, will serve as examples:

$$F = Eq = E'q'$$

$$F = Hm = H'm'$$

$$d\tau = dq/\rho = dq'/\rho'$$

$$c = D/E = D'/E'$$

$$\mu = B/H = B'/H'$$

$$t = q/I = q'/I'$$

$$F/L = IB = I'B'$$

$$dH/dt = RI^2 = R'I'^2$$

$$P = \Psi I = \Psi' I'$$

$$S = q/V, \quad S' = q'/V'$$

(but $S = \int DdS/V$ and $S' = \int D'dS/4\pi V'$)

$$P = \Phi/\Omega, \qquad P' = \Phi'/\Omega'$$

$$J = D_2 - c_1 E_2, \quad 4\pi J' = D_2 - c_1 E_2$$

$$J = B_2 - \mu_1 H_2, \quad 4\pi J' = B_2 - \mu_1 H_2$$

$$L = N/I, \qquad L' = N'/I'$$

etc., nearly all the equations being identical on the two systems.

On the other hand, while some of the derived equations are identical in the rational and irrational systems, many are not identical. Thus, for example, on developing the equations for the irrational systems, we find that

$$\Pi' = 4\pi q', \qquad \text{while } \Pi = q$$

$$S' = \Pi'/4\pi V', \quad \text{while } S = \Pi/V$$

$$V' = q'/cL, \qquad \text{while } V = q/4\pi cL$$

$$U' = \tfrac{1}{2}cE'^2/4\pi, \quad \text{while } U = \tfrac{1}{2}cE^2$$

$$T' = \tfrac{1}{2}\mu H'^2/4\pi, \quad \text{while } T = \tfrac{1}{2}\mu H^2$$

$$\text{curl } H' = 4\pi i', \quad \text{while curl } H = i$$

while also

$$W = \tfrac{1}{2}S'V'^2 = \tfrac{1}{2}q'^2/S' = \tfrac{1}{2}q'V' = \tfrac{1}{2}SV^2 = \tfrac{1}{2}q^2/S = \tfrac{1}{2}qV$$

$$W = \tfrac{1}{2}L'I'^2 = \tfrac{1}{2}LI^2$$

$$\text{curl } E' = -dB'/dt, \text{ curl } E = -dB/dt \qquad \text{etc., etc.}$$

The ratio of the rational electrostatic unit of a given quantity to the irrational electrostatic unit of the same quantity is always equal to the ratio of the corresponding rational electromagnetic unit to the irrational electromagnetic unit. This ratio for each of the principal electrical quantities is given in Table II.

TABLE II.—ELECTRICAL UNITS AND DIMENSIONS.

Unit.	Symbol.	Dimensions in Terms of [a] and [c]	Dimensions in Terms of [a] and [μ]	Rational Unit / Corresponding Irrational Unit	Electrostatic Unit / Electromagnetic Unit	Rational Electrostatic Unit / Irrational Electromagnetic Unit	Practical Unit / Irrational Electromagnetic Unit
Electric charge.	q	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	$a^{-1}(4\pi)^{-\frac{1}{2}}$	10^{-1}(Coulomb)
Electric volume density.	ρ	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{5}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	"	
Electric surface density.	σ	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	"	
Electric moment.	M	$[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	"	
Electric permittivity.	c	$[c]$	$[L^{-2}T^{2}\mu^{-1}a^{2}]$	1	a^{-2}	a^{-2}	
Electric intensity.	E	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}c^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}a^{-1}]$	$(4\pi)^{\frac{1}{2}}$	a	$a(4\pi)^{\frac{1}{2}}$	10^{8}(Volt per cm.)
Electric displacement.	D	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{\frac{1}{2}}$	a^{-1}	$a^{-1}(4\pi)^{\frac{1}{2}}$	
Electric flux	Π	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{\frac{1}{2}}$	a^{-1}	"	
Electromotive force.	V, Ψ	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}c^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}a^{-1}]$	$(4\pi)^{\frac{1}{2}}$	a	$a(4\pi)^{\frac{1}{2}}$	10^{8}(Volt)
Capacity.	S	$[Lc]$	$[L^{-1}T^{2}\mu^{-1}a^{2}]$	$(4\pi)^{-1}$	a^{-2}	$a^{-2}(4\pi)^{-1}$	10^{-9}(Farad)
Intensity of electrisation.	J	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	$a^{-1}(4\pi)^{-\frac{1}{2}}$	
Electric current.	I	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	"	10^{-1}(Ampère)
Electric current density.	i	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-2}c^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}\mu^{-\frac{1}{2}}a]$	$(4\pi)^{-\frac{1}{2}}$	a^{-1}	"	
Resistance.	R	$[L^{-1}T\ c^{-1}]$	$[L\ T^{-1}\mu a^{-2}]$	4π	a^{2}	$a^{2}(4\pi)$	10^{9}(Ohm)
Resistivity.	r	$[T\ c^{-1}]$	$[L^{2}T^{-1}\mu a^{-2}]$	4π	a^{2}	"	
Conductance.	K	$[L\ T^{-1}c]$	$[L^{-1}T\mu^{-1}a^{2}]$	$(4\pi)^{-1}$	a^{-2}	$a^{-2}(4\pi)^{-1}$	
Conductivity.	k	$[T^{-1}c]$	$[L^{-2}T\mu^{-1}a^{2}]$	$(4\pi)^{-1}$	a^{-2}	"	
Magnetic pole strength.	m	$[M^{\frac{1}{2}}L^{\frac{1}{2}}c^{-\frac{1}{2}}a]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]$	$(4\pi)^{-\frac{1}{2}}$	a	$a(4\pi)^{-\frac{1}{2}}$	
Magnetic moment.	M	$[M^{\frac{1}{2}}L^{\frac{3}{2}}c^{-\frac{1}{2}}a]$	$[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}]$	$(4\pi)^{-\frac{1}{2}}$	a	$a(4\pi)^{-\frac{1}{2}}$	
Magnetic inductivity.	μ	$[L^{-2}T^{2}c^{-1}a^{2}]$	$[\mu]$	1	a^{2}	a^{2}	
Magnetic intensity.	H	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}c^{\frac{1}{2}}a^{-1}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$	$(4\pi)^{\frac{1}{2}}$	a^{-1}	$a^{-1}(4\pi)^{\frac{1}{2}}$	
Magnetic induction.	B	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}c^{-\frac{1}{2}}a]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$	$(4\pi)^{\frac{1}{2}}$	a	$a(4\pi)^{\frac{1}{2}}$	
Magnetic flux.	Φ	$[M^{\frac{1}{2}}L^{\frac{1}{2}}c^{-\frac{1}{2}}a]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]$	$(4\pi)^{\frac{1}{2}}$	a	"	
Magnetomotive force.	Ω	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}c^{\frac{1}{2}}a^{-1}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$	$(4\pi)^{\frac{1}{2}}$	a^{-1}	$a^{-1}(4\pi)^{\frac{1}{2}}$	
Permeance.	P	$[L^{-1}T^{2}c^{-1}a^{2}]$	$[L\mu]$	1	a^{2}	a^{2}	
Intensity of magnetisation.	J	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}c^{-\frac{1}{2}}a]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$	$(4\pi)^{-\frac{1}{2}}$	a	$a(4\pi)^{-\frac{1}{2}}$	
Inductance.	I. & M	$[L^{-1}T^{2}c^{-1}a^{2}]$	$[L\mu]$	4π	a^{2}	$a^{2}(4\pi)$	10^{9}(Henry)

Practical unit of work = Joule = 10^7 ergs. Practical unit of activity = Watt = Joule per second = 10^7 ergs per second.

The ratio of the rational electrostatic unit to the rational electromagnetic unit of a given quantity is always equal to the ratio of the irrational electrostatic unit to the irrational electromagnetic unit of the same quantity. This ratio can easily be found in any case from the equations defining and connecting the units, and the relation given in § 4, XII., between the electromagnetic and electrostatic unit current. Thus the electromagnetic unit charge or current is a times as great as the corresponding electrostatic unit; the electrostatic unit magnetic pole strength is a times as great as the electromagnetic unit pole strength; the electromagnetic unit permittivity is a^2 times as great as the electrostatic unit permittivity; the electrostatic unit inductivity is a^2 times as great as the electromagnetic unit inductivity, etc., etc.; the ratio of the electrostatic to the electromagnetic unit being always a, a^2, a^{-1}, or a^{-2}.

The ratio of the electrostatic to the electromagnetic unit of each principal electrical quantity is given in Table II. The table also contains the ratio of the rational electrostatic unit of each quantity to the irrational electromagnetic unit.

As shown by the table, or by the preceding statement, the permittivity of free æther, which is unity on the electrostatic systems, is $1/a^2 \times$ unity on the electromagnetic systems. Also, the inductivity of free æther, which is unity on the electromagnetic systems, is $1/a^2 \times$ unity on the electrostatic systems. Thus the product of the permittivity of free æther by its inductivity, both measured on the same system of units, is equal, numerically, to $1/a^2$.

Of the dimensions (§ 6) of a nothing is known. In all that precedes we have *assumed* its dimensions to be zero, and we shall adhere to this assumption in what follows, that is we shall treat a as a mere number, except where the contrary is stated.

4. Experimental Determination of the Magnitude of a. To determine the value of a experimentally, it is necessary only to find the ratio between the electrostatic and electromagnetic measures

of any one electrical quantity. This will furnish a or a known power of a according to what precedes. Hence the ratio a can be determined in a great variety of ways. For example, the capacity of a standard condenser can be determined in absolute electrostatic measure from the measurement of its geometrical dimensions (§§1–2, III.), and can be determined in absolute electromagnetic measure by comparison with a resistance (§60, XIII.), or a mutual inductance (§58, XIII.), or other electrical quantity, independently determined in absolute measure according to the methods described above, or other methods. By such methods, and a number of others, some of which will be apparent from the methods of measurement previously discussed in this book, a has been determined with considerable precision. The best results all lie close (within a few tenths per cent.) to 3×10^{10}.

For some of the most recent and best determinations, see Thomson and Searle, *Phil. Trans.*, A, 1890; H. Abraham, *Ann. de Chim. et de Phys.*, Vol. 27, 1892; and D. Hermuzescu, *Ann. de Chim. et de Phys.*, Vol. 10, 1897.

According to the theory developed in Chapter XVI., electromagnetic waves are propagated in free æther with the velocity $v = 1/(c_0\mu_0)^{\frac{1}{2}}$, which is equal to a numerically. This velocity has been determined approximately for long waves, and very accurately for extremely short waves (light), and found to agree with a as otherwise determined within the limits of error of experiment.

5. Practical Units. The rational systems of units are at present unfortunately but little used, and the irrational electrostatic system is used only in pure science and mostly for theoretical purposes. The irrational electromagnetic system, on the other hand, is more extensively used. For the purposes of ordinary experimental work, however, especially for the purposes of electrical engineering, many of the electromagnetic units, as well as many of the c.g.s. mechanical units, are inconveniently large or small. Hence a practical system of units, each a decimal multiple or submultiple of the corresponding (irrational) *electromagnetic* unit,

or the c.g.s. mechanical unit, has been developed in which each of the units most frequently employed in practice has a magnitude of the same order as that of the quantities with which it is ordinarily to be compared. These units are so chosen as to form a self-consistent system satisfying all the equations satisfied by the electromagnetic system, except those equations which are not wholly made up of quantities whose units are defined in the practical system. The units of *mass*, *length*, *permittivity*, and *inductivity* are the same in both systems. The relations between the other practical units and the electromagnetic units are given in Table II. together with the names of the practical units.

A unit one million (10^6) times as great as any one of these units is designated by the name of the unit with the prefix *mega* or *meg*. Thus a *megohm* is one million *ohms*.

A unit one millionth (10^{-6}) as great as any one of these units is designated by the name of the unit with the prefix *micro*. Thus a *microvolt* or *microfarad* is one millionth of a *volt* or a *farad*.

In like manner the prefixes *deka*, *deci*, *centi*, etc., are attached to the names of the units with the same effects as they have upon the common units of the metric system.

6. The Dimensions of Electrical Quantities.* As already stated, nothing is known of the physical nature, or dimensions in mass, length, and time, of the quantities c, μ, and a. Hence, since

* **Vector Dimensions.** In the system of dimensions adopted here no account has been taken of the fact that a length, unlike a mass or a time, is a vector, or directed quantity. Thus, on this system, the dimensions of a plane angle, which is a length divided by a length, are zero in [M], [L], and [T], although one of the lengths is perpendicular to the other; the dimensions of a solid angle are zero, although it is a surface divided by the square of a length perpendicular thereto; the dimensions of work (force $\times$ distance in direction of force) are equal to the dimensions of torque (force $\times$ distance perpendicular to force), etc., etc. Yet there is an essential difference between a mere number, which of course has no dimensions, and a plane or solid angle, and there is an essential difference between the physical nature of a quantity of work and a torque.

These anomalies vanish, however, if we express all dimensions in terms of [M], [T], and [X], [Y], and [Z], three lengths at right angles to one another, thus taking account of the vector nature of L. On this system, a plane angle has such

every electrical unit involves one or more of these quantities, the complete dimensions of every electrical quantity are unknown. If, however, $[c]$, $[\mu]$, and $[a]$ are written for the unknown dimensions in mass, length, and time of c, μ, and a, respectively, a complete *expression* for the dimensions of every electrical quantity can be written down. Thus, if the dimensions of any quantity are obtained from the equations of either of the electrostatic systems, they will be expressed in terms of $[M]$, $[L]$, $[T]$, and $[a]$ and $[c]$; if they are obtained from the equations of either electromagnetic system, they will be expressed in terms of $[M]$, $[L]$, $[T]$, $[a]$, and $[\mu]$. The dimensions of all the principal electrical quantities, both in terms of $[a]$ and $[c]$ and in terms of $[a]$ and $[\mu]$, are given in Table II.

Since the actual dimensions in $[M]$, $[L]$, and $[T]$ of any electrical quantity must be the same whether expressed in terms of $[a]$ and $[c]$ or in terms of $[a]$ and $[\mu]$, the dimensions of any quantity in terms of $[a]$ and $[c]$ may be equated to its dimensions in terms of $[a]$ and $[\mu]$. Thus, equating the dimensions of electric charge in terms of $[a]$ and $[c]$ to its dimensions in terms of $[a]$ and $[\mu]$, we obtain

$$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}c^{\frac{1}{2}}] = [M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}a]$$

or

$$[a/c^{\frac{1}{2}}\mu^{\frac{1}{2}}] = [L/T] \tag{5}$$

That is, the quantity $a/c^{\frac{1}{2}}\mu^{\frac{1}{2}}$ is a *linear velocity*. Exactly the same relation, (5), and only this relation, follows from equating the two expressions for the dimensions of any other electrical quantity.

The velocity of electromagnetic waves as determined from the equations of Chapter XVI. is $1/(\mu c)^{\frac{1}{2}}$. In these equations, however, as stated above, the dimensions of a are ignored. If this is not done, it is easy to see that the velocity comes out equal to $1/(\mu c)^{\frac{1}{2}} \times [a]$, which is numerically equal to $1/(c\mu)^{\frac{1}{2}}$ and is dimensionally correct by (5).

dimensions as $[XY^{-1}]$ or $[YZ^{-1}]$; a solid angle such dimensions as $[XYZ^{-2}]$ or (YZX^{-2}); a quantity of work such dimensions as $[MX^2T^{-2}]$; a torque such dimensions as $[MXYT^{-2}]$; etc. This is therefore a *rational* system of dimensions. This system of dimensions, as applied to mechanical and electrical quantities, is discussed at length by W. Williams, in the *Philosophical Magazine*, September, 1892.

CHAPTER XV.

THE GENERAL ELECTRIC CURRENT.

1. Displacement Current and Magnetic Intensity. It has been shown in § 27, XIII., that when no other kind of current is present the conduction current across a surface is equal to the m.m.f. around the edge of the surface, and that the conduction current density is equal to the curl of the magnetic intensity.

That a changing electric displacement, or a *pure displacement current*, also gives rise to a magnetic field similar, qualitatively, to that of a conduction current Hertz proved by *direct* experiments. Consistently with these experiments, we shall here assume that a given displacement current develops a magnetic field similar, both qualitatively and quantitatively, to that connected with a conduction current of the same magnitude and distribution. The very important consequences of this assumption are in rigorous accord with experiment (XVI.). Thus we may write for a closed curve through which the electric flux is changing and through which there is no other form of current than a displacement current,

$$\Omega = I_d = d\Pi/dt \tag{1}$$

and $$\text{curl } H = i_d = dD/dt \tag{2}$$

which are analogous to (8) and (9), XIII.

Ω is called an *induced m.m.f.*, and H an *induced magnetic intensity*.

2. The Magnetic Field Induced by the Motion of a Concentrated Charge. Let an approximately concentrated charge $q = \rho d\tau$

move with a velocity v, Fig. 130. Let Π denote the electric flux through a circle of any radius a with its axis passing through $d\tau$ parallel to v, and let the direction of v be chosen as the positive direction through the circle. Owing to the motion of the charge with its field the flux through the circle will increase at the rate $d\Pi/dt$ and a m.m.f. equal to $\Omega = d\Pi/dt$ will be induced in the positive direction around the circle. By symmetry, the

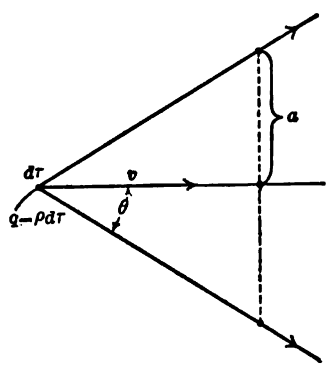


Fig. 130.

induced magnetic intensity H is equal in magnitude at all points of the circle which, like all other parallel circles centered on the same axis, is a line of magnetic intensity. Let θ denote the angle between v and the direction of D at every point of the circle; then

$$d\Pi/dt = v \cdot \sin\theta \cdot 2\pi a D = \Omega = 2\pi a H$$

and the directions of the vectors are so related that this equation gives

$$H = \mathrm{V}vD \cdot \sin\theta \qquad (3)$$

which is analogous to (5a), XIII.

Thus the magnetic intensity is developed by the motion of the tubes of electric displacement at right angles to their length.

Since in the case considered

$$D = q/4\pi r^2 = q\underline{r}/4\pi r^3$$

where $\underline{r}$ ($= r$ numerically) is the distance of the circle from $d\tau$, (3) is equivalent to

$$H = 1/4\pi r^3 \cdot \mathbf{V} vq \cdot \underline{r} \sin\theta = d\tau/4\pi r^3 \cdot \mathbf{V} \rho v \cdot \underline{r} \sin\theta$$
$$= d\tau/4\pi r^3 \cdot \mathbf{V} i_{cv} \underline{r} \sin\theta \quad (4)$$

Thus, as shown by a comparison of (4) with (13), XII., the same magnetic effects are produced by a moving charge with convection current density i_{cv} as by a conduction current of density $i_c = i_{cv}$.

In strictness the above results are only approximate and require an appreciable correction when v is comparable with the velocity of free electromagnetic disturbances (the velocity of light), § 11, XVI., since the field at a distance from a moving charge *lags* behind the charge.

A similar magnetic field would be developed by the motion of the pole of an electret, and an electric field exactly analogous to this last by the motion of the pole of a magnet, only the space outside the magnet or electret being considered.

3. The Magnetic Field of a Cylindrical Convection Current. The electric field of a cylindrical condenser is discussed in §§ 8–9, II. Let the charge upon unit length of the inner conductor be $+q$ and that upon unit length of the outer conductor therefore $-q$; and, for simplicity, suppose the conductivity of the outer cylinder *perfect*. Then the tubes of displacement will terminate normally upon the outer cylinder whether the inner cylinder is at rest or in motion. Let the inner cylinder move with a velocity v in the direction AB of its axis. Then the convection current through any closed curve surrounding the inner cylinder is

$$I_{cv} = qv$$

in the direction AB.

By the last article, the m.m.f. around any closed curve surrounding the inner cylinder and within the outer cylinder is therefore

$$\Omega = I_{cv} = qv$$

The magnetic intensity has the same magnitude at all points of any circle of radius r coaxial with the cylinders, and all such circles are lines of magnetic intensity. Hence

$$\Omega = 2\pi r H = qv = 2\pi r D v$$

and, with due respect to the directions of the vectors,

$$H = \mathbf{V} v D \tag{5}$$

In this case there is no displacement current through the circle of intensity, but the magnetic field is developed as before by the motion of tubes of displacement perpendicularly to their length. (5) is a particular case of (3), since in this case $\sin\theta = 1$.

4. Experiments upon the Convection Current. That an electric convection current, in conformity with the above theory, is accompanied by a magnetic field of the same character, both qualitatively and quantitatively, as that connected with a pure conduction current of the same magnitude and distribution, has been proved in several series of experiments by Rowland, using charged rotating discs, and has been confirmed by many others.

Just as the magnetic field of a conduction current may be deduced as a consequence of Ampère's law, § 11, XII., so, conversely, Ampère's law may be deduced as a consequence of the (experimentally investigated) magnetic field of the current. Since therefore a convection current develops a magnetic field identical with that of a conduction current of the same magnitude and distribution, Ampère's law must apply to such a current. Hence a beam of cathode rays (consisting of very fine negatively charged particles, or *electrons*, moving with velocities approaching that of light) should be deflected when immersed in a magnetic field perpendicular to the beam. That such a deflection occurs, in qualitative agreement with the theory, has been known for many years. A deflection in a magnetic field, in the direction indicated by theory, of the much more massive and more slowly moving positively charged particles forming the *canal* rays has also been

recently observed. The assumption, justified by the experiments of Rowland, that the agreement between these experiments and theory is also quantitative has recently led to very important advances in knowledge.

In accordance with what precedes, we may write for the m.m.f. around the edge of a surface across which there is a convection current only

$$\Omega = I_{cv} \tag{6}$$

and

$$\text{curl } H = i_{cv} \tag{7}$$

5. Magnetic Intensity and the Motion of Electric Displacement. Motional Magnetic Intensity. In § 2 it has been shown that for the case there considered (1), and therefore (2), is equivalent to (3). In the same way, just as (8) and (9), XIII., are equivalent to (5*a*), XIII., so (1) and (2) may be readily shown to be equivalent, in the general case, to the more general relation

$$H = \mathbf{V}vD \sin \theta \tag{8}$$

where H is the magnetic intensity at a point P in the dielectric, at which the displacement is D, developed by the motion of the tubes of displacement relatively to the medium at P with the velocity v, whose component perpendicular to D is $v \sin \theta$.

If the medium at P is in actual motion with reference to the surrounding medium, containing the inducing system, such as fixed charges, it is an *intrinsic* magnetic intensity, and is called a *motional* magnetic intensity. The existence of a motional magnetic intensity was first established by Roentgen, in whose experiments a magnetic field in qualitative agreement with (8), to judge from its continuation in the air, was developed in a slab of rigid dielectric rotated in air between fixed charged discs. For the most recent experiments upon the subject, see A. Eichenwald, *Ann. der Physik*, Nos. 5 and 6, 1903. The experiments indicate that for D, in (8), if c_2 and c_1 ($= c_0$, sensibly) denote the permittivities of the slab and the air, respectively,

$$J = [(c_2 - c_1)/c_2]D$$

should be substituted. That is, a *fictitious* convection current produces the same magnetic effects as a true convection current of the same magnitude and distribution.

Thus magnetic intensity is induced by moving tubes of electric displacement, as electric intensity is induced by moving tubes of magnetic induction (§ 6, XIII.). It cannot be said, however, that magnetic intensity is always due to moving tubes of electric displacement, or that electric intensity is always due to moving tubes of magnetic induction, an attempt to make (5*a*), XIII., and (8) general expressions for magnetic and electric intensity leading to apparently insurmountable difficulties. It is sufficient to mention the field of a static electric charge or magnetic pole.

6. (Induced?) Intensities in the Field of a Steady Conduction Current. Let the current I_c traverse the inner and outer conductors of a cylindrical condenser axially in opposite directions, the inner cylinder in the direction AB, and suppose both cylinders perfect conductors. Then the electric field will be radial and perpendicular to both cylinders as in a purely static field.

Imagine the conduction current to consist in the motion of the positive and negative ends of the electric tubes along AB and CC respectively, the electric tubes travelling in the direction AB with the velocity v. Then the magnetomotive force around a circle of magnetic intensity of radius r is

$$\Omega = I_c = 2\pi r H = qv = 2\pi r D v$$

q being the charge upon unit length of the inner cylinder and D the displacement at the distance r from the axis. With due respect to the signs of the vectors, the last equation gives

$$H = \mathbf{V} v D \tag{9}$$

Thus by *assuming* that the magnetic field of the steady conduction current is a consequence of the (*assumed*) motion of its electric field we are led to the same relation between H, v and D as already deduced for the displacement current.

If we *imagine* the electric field, likewise, to be developed by the (*assumed*) motion of the magnetic field with velocity u, (5*a*), XIII., gives

$$E = \mathbf{V}Bu \tag{10}$$

Equations (9) and (10) give for the relations between u and v

$$uv = 1/c\mu$$

$$u/v = \tfrac{1}{2}ED/\tfrac{1}{2}HB$$

$$u\tfrac{1}{2}HB = v\tfrac{1}{2}ED = \tfrac{1}{2}EH$$

Since the electric and magnetic energy densities are not in general equal, except in the case of pure electromagnetic waves (XVI.), the above conceptions lead to the anomalous result that u and v are different and may have *any* ratio to one another. Since in some cases (according to the dissociation theory), and probably in all (§ 15, IX.), the steady electric current in an actual conductor consists in the motion throughout the conductor in opposite directions of positively and negatively charged particles, and since the surface charges connected with the external field of the steady current do not take part in the conduction (§ 9, VIII.), the above results must be taken as at present only suggestive.

7. The First Law of Circuitation for Media at Rest and in Motion. When a given surface in a medium at rest is crossed by conduction, convection, and displacement currents simultaneously, the total current I through the surface and the m.m.f. Ω around its edge (the direction of the m.m.f. being related to that of the current as the direction of rotation to that of translation of a right-handed screw) are connected by the equation

$$I = I_c + I_d + I_{cv} = \Omega \tag{11}$$

and the total current density i and the magnetic intensity H are connected by the equation

$$i = i_c + i_d + i_{cv} = \text{curl } H \tag{12}$$

where $i_e + i_d + i_{cv}$ is a vector sum. Here H denotes either the intensity due to the currents, or the total intensity due to both currents and magnets, if such are present (since the curl of the intensity due to the poles of a magnet is zero).

If the medium is in motion, with the velocity $-v$ at the point where the electric displacement is D (§ 5), and if H still denotes the total magnetic intensity, the curl of the motional magnetic intensity, which we shall call the *motional current density*, or the *fictitious convection current density*, and denote by i_m, must be added to the first member of (12). Thus we have,

$$i = i_e + i_d + i_{cv} + i_m = \text{curl } H \qquad (13)$$

the first member being a vector sum. This is the most general form of the first law of circuitation.

8. The Circuital Character of the Total Current. Kirchhoff's Law I. Generalised. That a steady conduction current flows in a closed circuit (div $i_c = 0$ everywhere) is shown in § 5, VIII.

From the Cartesian expression for the divergence of a vector (§ 31, I.), and the Cartesian expression for the curl of a vector (§ 4, XVI.), it follows that the divergence of the curl of any vector is zero. Hence it follows from (12) and (13), since i, the total current density, is equal to curl H, that

$$\text{div } i = \text{div curl } H = 0 \qquad (14)$$

That is, in any case, the *total* current flows in closed tubes.

Thus, for example, let an electric condenser AB be discharged through a wire C, and first suppose the capacity of the wire negligible in comparison with that of the rest of the system. Then during the discharge the conduction current $I = -dq/dt$ will be sensibly the same across every section of the wire (if the capacity of the wire were zero, any charge there accumulating would produce an infinite potential difference), and the displacement and displacement current will be confined sensibly to the region occupied by the tubes stretching from A to B. During

the discharge the flux from A to B decreases at the rate $-dq/dt$ and the flux from B to A increases at the same rate $-dq/dt$. Thus the displacement current $d\Pi/dt = -dq/dt$ from B to A through the dielectric across any closed surface surrounding one of the plates is equal to the conduction current across every section of the wire from A to B. Thus the total current flows in a closed circuit, the displacement current starting where the conduction current stops.

Similar phenomena occur when the wire is cut and its ends connected to the terminals of a voltaic cell. The conduction current through the wire and cell as the condenser is charged is equal to the displacement current in the same direction around the circuit through the dielectric.

If the capacity of the wire C differs from zero (which is always the case to a greater or less extent), then, if the condenser is discharged by bringing the wire into contact with the plates at A and B simultaneously, tubes of displacement will move out along the wire stretching from the part near A to the part near B, giving rise to a displacement current through the dielectric from the one part to the other as well as to a conduction current through the wire; and the sum of the two currents, through the wire and through the external dielectric from A to B, is equal to the displacement current from B to A through the rest of the dielectric. During this process the conduction current is not constant from section to section of the wire, being zero across the more remote parts of the wire while it has an appreciable value across the nearer parts immediately after the beginning of the discharge.

CHAPTER XVI.

THE TRANSFERENCE OF ELECTROMAGNETIC ENERGY. ELECTROMAGNETIC WAVES. MAXWELL'S THEORY.

1. Poynting's Theorem * when E is Perpendicular to H. Fig. 131 represents one end of a system consisting of two long coaxial perfectly conducting circular cylinders A and F, with external and internal radii R_1 and R_2 respectively, closed by a non-conducting slab E of zero permittivity, and plugged with a closely fitting right cylindrical conductor D of length L and resistance

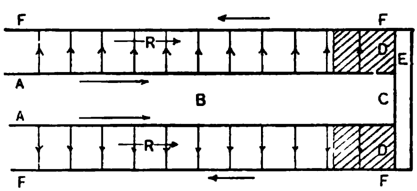


Fig. 131.

W. A and F are connected at the remote end of the system with the positive and negative terminals of a voltaic cell or other electric generator, and are traversed by a constant current in the direction $ABCDFF$.

Since A and F are perfect conductors and since E has zero permittivity, the electric and magnetic fields are confined wholly to the dielectric and to the conductor D. (If the permittivity of E were not zero, the fields would be the same except that in addition a static *electric* field would extend beyond C to the right connecting F and C.) Also, the lines of electric intensity are

* J. H. Poynting, *Phil. Trans.*, Part II., 1884 and Part II., 1885; Oliver Heaviside, *Electrical Papers* and *Electromagnetic Theory*; J. H. Poynting, *Rapports*, Congrès Int. de Physique, 1900, Vol. II.

normal to both cylinders and straight, as in a purely static field. The lines of magnetic intensity are circles centered on the axis of the cylinders in planes perpendicular thereto.

Let the steady current be denoted by I, and the steady difference of potential from A to F (there is no fall of potential along A or along F) by $V = WI$.

Energy, generated by the voltaic cell (or other generator), is supplied to the system ABF at the rate VI, and is dissipated by the resistance W at the rate $WI^2 = VI$. Since there is no energy within the conductors A and F or outside the closed system, energy must therefore be *transferred* in the direction ABC across every section of the *dielectric* at the rate

$$(R) = VI = WI^2 \tag{1}$$

Since $V = \int Edr$ along a line of electric intensity, and $I = 2\pi rH$, where E and H denote the electric and magnetic intensities at a circle in the dielectric distant r from the axis, (1) may be written

$$(R) = VI = \int EH \cdot 2\pi rdr = \int EHdS$$

where S denotes the cross-section $\int 2\pi rdr$ of the dielectric. Therefore, due attention being paid to the signs of the vectors, the time rate per unit area, $\mathbf{R}$, at which electromagnetic energy is transferred across an area whose plane contains E and H, perpendicular to one another, or the *electromagnetic energy flux density*, is

$$\mathbf{R} = d(R)/dS = \mathbf{V}EH \tag{2}$$

This is *Poynting's theorem* for the case in which E is perpendicular to H. A more general form of the theorem will be developed in § 5.

Let L and S denote the inductance and capacity of unit length of the system, then $\frac{1}{2}LI^2$ and $\frac{1}{2}SV^2$ denote the magnetic and electric energies contained in a unit length of the dielectric. Let v denote the velocity with which the electric energy moves in the direction AB, and u the velocity with which the magnetic

energy moves in the same direction (see § 6, XV.). Then we may write

$$(R) = v(\tfrac{1}{2}SV^2) + u(\tfrac{1}{2}LI^2) \tag{3}$$

which is equal to VI if

$$uv = 1/SL = 1/c\mu \text{ (see below)} \tag{4}$$

In exactly the same way we have for the energy flux density

$$\mathbf{R} = v(\tfrac{1}{2}cE^2) + u(\tfrac{1}{2}\mu H^2) \tag{5}$$

which is equal to EH if $uv = 1/c\mu$.

Thus the conception of moving tubes is consistent with Poynting's theorem if the relation (4) holds between the velocities.

When $\tfrac{1}{2}LI^2 = \tfrac{1}{2}SV^2$, or $\tfrac{1}{2}cE^2 = \tfrac{1}{2}\mu H^2$,

$$u = v = 1/(c\mu)^{\frac{1}{2}} \tag{6}$$

2. Mechanical Analogue. Consider a circular cylindrical rod rotating uniformly about its axis AB and transmitting power in the direction AB. Let the constant angular velocity be denoted by I and the torque acting across every section by V, the common direction of both being related to the direction AB as the direction of rotation to that of translation of a right-handed screw.

The rate at which energy is transferred across every section of the rod in the direction AB is

$$(R) = VI$$

Owing to the torque across every section, the rod is *twisted*, or any two sections are *sheared* with respect to one another.

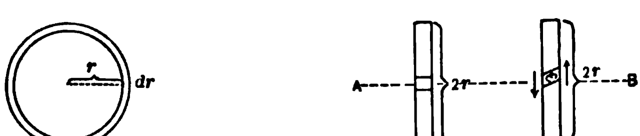


Fig. 132.

For all points at a given distance r from the axis the (angle of) shear, or relative shift per unit length between two cross-sections,

is the same and will be denoted by D. Fig. 132 shows the natural and sheared states of a ring of radius r and infinitesimal thickness dr. D is zero at the axis and is proportional to r, having its greatest value at the surface of the rod. The direction of D is everywhere radial from the axis, since a right-handed screw at any point rotating in the direction of the twist or shear at the point, as shown by the arrows in the figure, would move radially toward the surface.

Let the modulus of rigidity of the rod be denoted by n, and its reciprocal, or the shear permittivity, by c. Then, at any point of the ring considered, the shearing stress (shearing force per unit area, or shearing torque per unit volume), which will be denoted by E, has the same direction as that of D and is equal to

$$E = nD = D/c$$

Since the area of the cross-section of a ring of radius r and thickness dr is $2\pi r dr$, the torque about the axis acting upon the cross-section is

$$dV = 2\pi r dr E r$$

Let the linear velocity at any point distant r from the axis be denoted by H. Then

$$H = rI$$

The rate at which energy is transferred across the zone of radius r and width dr is

$$IdV = 2\pi r^2 dr EH/r = EH 2\pi r dr$$

and the rate of transfer per unit area is

$$\mathbf{R} = \mathbf{V} EH \tag{a}$$

if due attention is paid to signs.

The potential energy per unit volume is $\frac{1}{2}cE^2 = \frac{1}{2}nD^2$.

Let the density of the rod be denoted by μ. Then the kinetic energy per unit volume is $\frac{1}{2}\mu H^2$.

If we *assume* the potential energy to travel in the direction AB

with the velocity v and the kinetic energy with the velocity u, we have also

$$\mathbf{R} = v\tfrac{1}{2}cE^2 + u\tfrac{1}{2}\mu H^2$$

which is consistent with (a) if $uv = 1/c\mu$. If $u = v$

$$\mathbf{R} = v(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2)$$

which, combined with (a), gives

$$v = \mathbf{V}EH/(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2)$$

3. Two Perfectly Conducting Parallel Circular Plates Connected by a Right Circular Coaxial Conducting Cylinder. Let L, Fig. 133, denote the distance between the plates, or the length of the

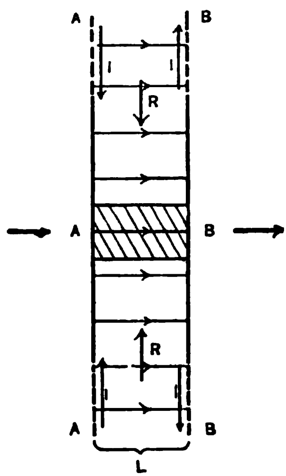


Fig. 133.

cylinder, a the radius of the cylinder, both supposed small in comparison with the radius of the plates, and W the resistance of the cylinder. Let the plates be maintained at the constant difference of potential V.

The electric field between the plates (except near their edges) is uniform and parallel to the axis of the cylinder in the direction AB. The electric intensity is $E = V/L (= D/c$ in the dielectric).

The current flows in radial stream-lines in the plane AA toward the axis of the cylinder, through the cylinder parallel to its axis in the direction AB, and from the cylinder in radial stream lines in the plane BB.

The lines of magnetic intensity are circles parallel to the planes and centered on the axis of the cylinder. The intensity at all points of a circle of radius r is $H = I/2\pi r = V/2\pi r W$, if r is greater than a. If r is less than a, $H = rI/2\pi a^2$.

Within the perfect conductors there is no field of either kind.

Energy is dissipated in the cylinder at the rate $(R) = WI^2 = VI$. Hence energy is *transferred inwardly* across every cylindrical surface with its ends in the planes A and B and enclosing the cylinder at the same rate

$$(R) = VI \tag{a}$$

If the surface is a right circular cylinder of radius r coaxial with the conducting cylinder, (a) becomes

$$(R) = EL \times 2\pi r \times H = EH \cdot 2\pi r L$$

But $2\pi r L$ is the area of the surface considered. Hence the energy flux density is, if due attention is paid to signs,

$$\mathbf{R} = \mathbf{V}EH \tag{b}$$

as already proved for perpendicular intensities in § 1.

The energy contained in each tube of displacement per unit cross-section is $\frac{1}{2}VD$. The velocity of the tubes of displacement (see § 6, XV.), or of the electric energy, inward at the distance r (greater than a) from the axis, if we *assume* that (8), § 5, XV., applies to the field of the steady conduction current, is

$$v = H/D = L/2\pi c r W \tag{c}$$

Hence the rate at which electric energy crosses the surface inward, or the rate at which electric energy is dissipated in the conductor, is, according to the conception of moving tubes,

$$\tfrac{1}{2}VD \times 2\pi r \times v = \tfrac{1}{2}WI^2$$

The magnetic energy density is $\frac{1}{2}\mu H^2$. The velocity of the magnetic tubes at the distance r (greater than a) from the axis is

$$u = E/B = 2\pi\mu r W \tag{d}$$

if we assume that (8), XV., and (6), XIII., apply to the field of the steady conduction current.

Hence the rate at which magnetic energy crosses the surface inward, or the rate at which magnetic energy is dissipated in heat in the conductor, is, according to the conception of moving tubes,

$$\tfrac{1}{2}\mu H^2 \times 2\pi r \times u = \tfrac{1}{2} W I^2$$

Thus the energy dissipated by resistance is, according to this conception, half electric and half magnetic. The same thing may be shown to be true in the system considered in § 1, the velocities of the electric and magnetic tubes being there, as here, inversely proportional to the corresponding energy densities ($u/v = \frac{1}{2}cE^2/\frac{1}{2}\mu H^2$).

Within the cylinder the electric intensity is constant, the number of tubes entering per second being equal to the rate at which tubes are broken up. The magnetic tubes contract as they approach the axis, thus giving up their energy without being broken up; and as the magnetic intensity decreases the velocity of the tubes increases in such a way as to make the number of unit tubes cutting unit length of a line of electric intensity per second constant ($E = Bu =$ constant). This appears also from the relation

$$u = E/B = 2\pi a^2 W/\mu r L \tag{e}$$

Inside the conductor (c) is of course unmeaning.

On multiplying together (c) and (d), we see that

$$uv = 1/c\mu$$

as in § 1.

4. The Cartesian Expressions for the Rectangular Components of the Curl of a Vector. Curl H and Curl E. The X, Y, Z components of a vector H being denoted by H_1, H_2, H_3, the X, Y, Z

components of its curl will be denoted by $\text{curl}_1 H$, $\text{curl}_2 H$, $\text{curl}_3 H$, respectively.

To find the Cartesian expression for $\text{curl}_1 H$ at a point $P(x, y, z)$, or 1, Fig. 8, at which the components of H are H_1, H_2, and H_3, we may take the line integral of H around the infinitesimal rectangle 17, the plane of which is parallel to the YZ plane and the sides of which, of lengths dz and dy, are parallel to Z and Y, divide the result by the area of the rectangle, $dS_1 = dydz$, and pass to the limit.

This integration, which must be performed in the direction of the arrows around the circuit, gives

$$\begin{aligned} d\Omega_1 = (H_2 + \tfrac{1}{2}dH_2/dy\,dy)dy + [H_3 + dH_3/dy\,dy \\ + \tfrac{1}{2}d(H_3 + dH_3/dy\,dy)/dz\,dz]dz - [H_2 + dH_2/dz\,dz \\ + \tfrac{1}{2}d(H_2 + dH_2/dz\,dz)/dy\,dy]dy - (H_3 + \tfrac{1}{2}dH_3/dz\,dz)dz \end{aligned}$$

the separate terms being the integrals along the sides 1, 2, 3, 4 in the order given. Cancelling equal and opposite terms, dividing by dS_1, and passing to the limit, we obtain

$$d\Omega_1/dS_1 = \text{curl}_1 H = dH_3/dy - dH_2/dz \qquad (a)$$

By an exactly analogous process, or by the principle of symmetry and inspection of (a), we find

$$\text{curl}_2 H = dH_1/dz - dH_3/dx \qquad (b)$$

and

$$\text{curl}_3 H = dH_2/dx - dH_1/dy \qquad (c) \qquad (7)$$

From these equations we may write down at once the components of the curl of any other vector E. Thus

$$\left.\begin{aligned} \text{curl}_1 E &= d\,E_3/dy - d\,E_2/dz & (a) \\ \text{curl}_2 E &= d\,E_1/dz - d\,E_3/dx & (b) \\ \text{curl}_3 E &= d\,E_2/dx - d\,E_1/dy & (c) \end{aligned}\right\} \qquad (8)$$

5. The Flux of Electromagnetic Energy. Poynting's Theorem. Let **R** denote, in both magnitude and direction, the time rate per

unit area at which electromagnetic energy is transferred at a point P (x, y, z) across a surface normal to the direction of transfer. Let the direction cosines of $\mathbf{R}$, E, and H at the point P be denoted by $l, m, n, l', m', n', l'', m'', n''$, respectively; and let the angle between E and H be denoted by θ. E and H will be used to denote the non-intrinsic intensities of the field, intrinsic intensities, when present, being denoted by e and h. With these conventions we have (*Poynting's theorem*)

$$\mathbf{R} = \mathbf{V}EH \sin\theta \tag{9}$$

to the demonstration of which we proceed.

Consider first a region containing no intrinsic electric or magnetic intensity. From the definition of $\mathbf{R}$ and that of the convergence of a vector it is evident that the rate at which electromagnetic energy enters an infinitesimal volume $d\tau$ at P through its surface (minus the rate at which energy leaves the volume) is conv $\mathbf{R} \cdot d\tau$. Hence, since no electromagnetic energy is developed within $d\tau$ $(e = h = 0)$, this quantity is equal to the rate of increase of electromagnetic energy plus the rate of dissipation of energy in heat within the volume. That is

$$\text{conv } \mathbf{R} \cdot d\tau = d\tau \cdot d(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2)/dt + d\tau \cdot kE^2$$

or

$$\begin{aligned}\text{conv } \mathbf{R} = d(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2)/dt + kE^2 = c(E_1 dE_1/dt \\ + E_2 dE_2/dt + E_3 dE_3/dt) + \mu(H_1 dH_1/dt \\ + H_2 dH_2/dt + H_3 dH_3/dt) + E_1 i_{c_1} + E_2 i_{c_2} + E_3 i_{c_3}\end{aligned} \tag{a}$$

Since

$$-\mu dH_1/dt = \text{curl}_1 E, \text{ etc.}$$

and

$$c dE_1/dt + i_{c_1} = \text{curl}_1 H, \text{ etc. (since } i_{cv} = i_m = 0)$$

(9) may be written

$$\begin{aligned}\text{conv } \mathbf{R} = E_1 \text{ curl}_1 H + E_2 \text{ curl}_2 H + E_3 \text{ curl}_3 H - H_1 \text{ curl}_1 E \\ - H_2 \text{ curl}_2 E - H_3 \text{ curl}_3 E = -[d(E_2 H_3 - E_3 H_2)/dx \\ + d(E_3 H_1 - E_1 H_3)/dy + d(E_1 H_2 - E_2 H_1)/dz]\end{aligned} \tag{b}$$

But also, by definition of the convergence of a vector,

$$\text{conv } \mathbf{R} = -(d\mathbf{R}_1/dx + d\mathbf{R}_2/dy + d\mathbf{R}_3/dz)$$

Hence $$\mathbf{R}_1 - a_1 = E_2H_3 - E_3H_2 = EH(m'n'' - n'm'')$$

$$\mathbf{R}_2 - a_2 = E_3H_1 - E_1H_3 = EH(n'l'' - l'n'')$$

$$\mathbf{R}_3 - a_3 = E_1H_2 - E_2H_1 = EH(l'm'' - m'l'')$$

where a, with components a_1, a_2, a_3, is a vector whose convergence is zero.

Without affecting in any way the generality of the conclusions, we may, for simplicity, give the rectangular coördinate system such an orientation as to make the plane XY parallel with the plane containing E and H, and the direction of X coincident with that of E. Then $E_1 = E$, $E_2 = E_3 = 0$, or $l' = l$, $m' = n' = 0$; and $H_3 = 0$, or $n'' = 0$, while $l'' = \cos\theta$ and $m'' = \cos(90° - \theta) = \sin\theta$. With these simplifications the above equations become

$$\mathbf{R}_1 - a_1 = 0 = \mathbf{R}_2 - a_2, \text{ and } \mathbf{R}_3 - a_3 = \mathbf{R} - a = EH\sin\theta$$

Hence, with due respect to the directions of the vectors,

$$\mathbf{R} = \mathbf{V}EH\sin\theta + a \tag{10}$$

Since div a = conv $a = 0$, a, if it is not zero, represents a flux of energy in *closed tubes* and therefore contributes nothing to the net energy entering any volume. In what follows this circuitous energy flux will be neglected, or, what amounts to the same thing, a will be assumed equal to zero, unless the contrary is stated. With this assumption, (10) is identical with (9).

If at any point P there is an intrinsic electric intensity e and an intrinsic magnetic intensity h in addition to the field intensities E and H, then an element of volume at P, in addition to receiving electromagnetic energy by *transference* across its surface at the time rate conv $\mathbf{R}$ per unit volume or $\mathbf{R}$ per unit area, receives electromagnetic energy by *transformation* on the spot at the time rate

$$ei\cos\theta' + hdB/dt\cdot\cos\theta''$$

per unit volume, where θ' denotes the angle between e and i, and θ'' that between h and dB/dt.

The vector **R** is called, as stated in § 1, the *electromagnetic energy flux density.*

Since **R** is perpendicular to E and to H, the lines along which the energy flows, or the *energy stream-lines*, are the intersections of the electric and magnetic equipotential surfaces. (Even when the field is not static or steady, so that the term potential cannot be used legitimately, we may still use the expression equipotential surface to denote a surface perpendicular at every point to the intensity.)

6. A Long Circular Cylindrical Conductor Traversed by a Steady Current. Fig. 134 shows a section of a small part of the electric field within and without the conductor when the current has the

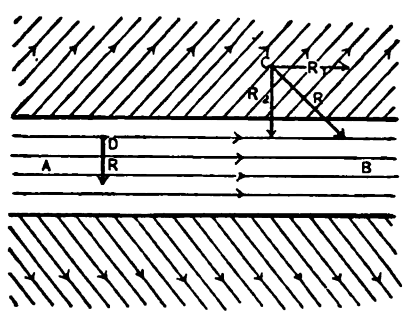


Fig. 134.

direction AB. The lines of magnetic intensity are circles about the axis AB of the cylinder, going down into the paper below AB and coming up out of the paper above AB. The electric and magnetic intensities are everywhere perpendicular.

Outside the wire, the energy flux density **R** has the direction indicated at C, with a component $\mathbf{R}_1$ in the direction of the axis AB, and a component $\mathbf{R}_2$ toward the axis. Thus in the dielectric energy is transferred in the direction AB to parts of the field farther along the circuit by the component $\mathbf{R}_1$, and energy is also

transferred toward and into the conductor by the component $\mathbf{R}_2$. Within the conductor, as at D, $\mathbf{R}$ is directed toward the axis with no component parallel to the axis. Hence within the conductor there is no transfer of energy along the circuit, all the energy entering the conductor from the dielectric and being dissipated in heat by resistance.

If a conductor, either insulated from the circuit or connected thereto at a single point, is placed in the electromagnetic field, then there will be a magnetic field inside the conductor, but no electric field, and the tubes of electric displacement will terminate normally upon the outer surface of the conductor. Within the conductor there is no transference of energy, since E is zero. In the dielectric at the surface of the conductor the energy stream-lines are parallel to its surface, since they are perpendicular to E, which is normal to the surface. Thus the energy streams around the conductor as a liquid streams around an impervious solid.

7. The Transfer of Energy in and About a Voltaic Cell and a Simple Electrolytic Cell. Figs. 135–138 represent diagrammatically for several cases the electric field and the transfer of electromagnetic energy in and about a Daniell cell under the assumptions (for which evidence, though not wholly satisfactory, can be adduced) that the single difference of potential from the copper electrode to the copper sulphate solution is positive and equal to that from the zinc sulphate solution to the zinc, and that the single difference of potential from the copper sulphate solution to the zinc sulphate solution is negligible. ABC represents the copper electrode, HIJ the zinc electrode, $DEFG$ the solutions, and the dotted line the porous partition between them. The distance between the electrolyte and the electrodes is of course enormously exaggerated in the diagrams. The intrinsic electromotive forces are directed from the copper sulphate solution to the copper and from the zinc to the zinc sulphate solution exactly opposite to the electric fields they develop.

The electric field on open circuit is shown in Fig. 135. There is no electric intensity within the conductors, no current, no magnetic field, and no transfer of energy.

The field after closing the circuit above *A* and *H* is shown in Fig. 136. Above *BI* the magnetic intensity is directed (in the plane of the diagram) normally into the paper, while below *BI* it is directed up out of the paper. The direction of the transference of electromagnetic energy is shown by the arrows cutting

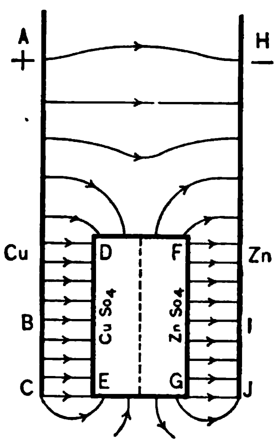


Fig. 135.

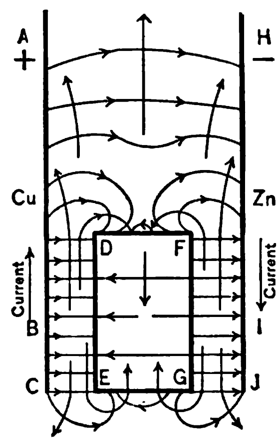


Fig. 136.

the lines of intensity normally. Electromagnetic energy, transformed from chemical energy with the deposition of copper and the solution of zinc at the electrodes, moves out from between the electrolyte and the electrodes into the dielectric, part there converging upon and moving into the electrolytic and metallic conductors, there to be dissipated in Joulean heat, and part being carried into other parts of the field.

The field and transference of energy when an agent with a considerably greater e.m.f. than that of the given cell sends a current through it (or assists in so doing) in the same direction as before is shown in Fig. 137. Electrical energy generated by the

external e.m.f., as well as all the energy generated by the intrinsic e.m.f. of the cell itself, is dissipated in the conductors.

The field and the transference of energy when a current is sent through the cell in opposition to its intrinsic e.m.f., that is from copper to zinc, is represented in Fig. 138. Here a portion of the energy supplied to the cell by the external e.m.f. is dissipated in the conductors and the rest is transformed into chemical energy with the deposition of zinc at the kathode and the solution of copper at the anode.

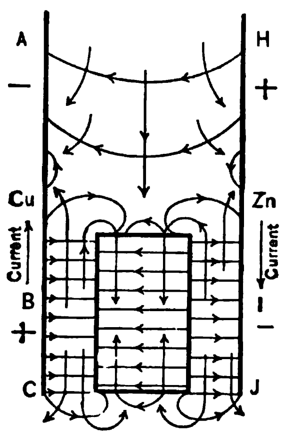


Fig. 137.

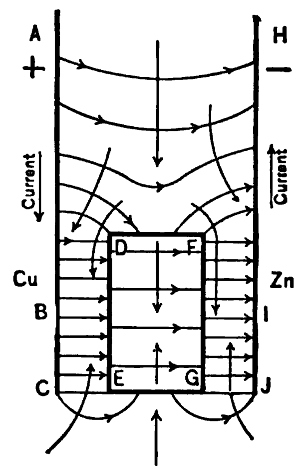


Fig. 138.

In Fig. 139 the electric field and the transference of energy in and about an electrolytic cell, consisting of copper electrodes *AC* and *HJ* dipping in a solution *DEFG* of copper sulphate, are represented diagrammatically for the case in which a current traverses the system from *A* to *H*. Between the kathode *HJ* and the electrolyte energy is transformed from chemical to electrical with deposition of copper, whence it moves out into the dielectric, and thence partly into the conductors and partly into the region between the anode *AC* and the electrolyte, where reconversion into chemical energy occurs. The quantity of chemical energy transformed into electrical at the kathode is equal to the quantity of electrical energy transformed into chemical energy

at the anode. Hence, since a portion of the electrical energy generated at the kathode is dissipated in heat, as much electrical energy coming from the external e.m.f. producing the current is transferred into the region about the anode, and there converted

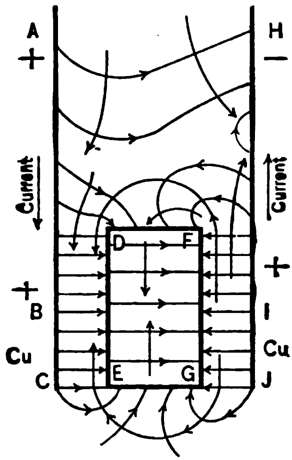


Fig. 139.

into chemical energy, as is dissipated in the conductors of the electrical energy generated at the kathode. On the whole, no work is done in the cell except that done upon resistance. (If the electrodes are at different levels, work will be done by or against gravity during the conduction, etc.)

8. A Circuit Containing a Motional E.M.F. We shall consider only the simple case of a slider, *AB*, Fig. 140, running on

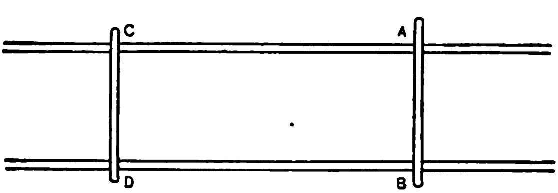


Fig. 140.

two parallel rails *AC* and *BD*, connected by a cross piece *CD* and immersed in a uniform magnetic field directed downward into the

plane of the paper. Let AB move to the right. Then there is induced in AB an intrinsic e.m.f. in the direction BA, producing a current around the circuit in the direction $BACDB$, with lines of magnetic intensity related to the current in the usual manner. The intrinsic e.m.f. in the direction BA produces an electric field intensity with the general direction from A to B along the slider and around the circuit, both within and without the conductor. The lines of electric intensity diverge from the upper half of AB in the dielectric and converge upon the lower half. Hence, since $\mathbf{R} = \mathbf{V}EH \sin\theta$, the electromagnetic energy generated in AB moves *outwards* from AB and toward its ends, then contracts upon the rest of the circuit, all the energy being finally dissipated in heat. While the energy is moving outward through the conductor AB a part of the energy is dissipated owing to the resistance of AB, not all the energy developed in AB reaching the dielectric.

9. A Concentrated Electric Charge in the Presence of a Concentrated Magnetic Pole. If the charge and pole are concentrated at two points A and B, respectively, and if the energy flux density is given by $\mathbf{V}EH \sin\theta$, the energy stream-lines are circles about AB and AB produced as axis. Since both fields are purely static in this case, however, there is no reason to believe that any flow of energy, even in closed tubes, exists. To reconcile this view with Poynting's theorem, we have only to remember that the energy flux density, in the general case, is $\mathbf{V}EH \sin\theta$ *plus* a circuital flux density a, and to suppose that in the present case $a = -\mathbf{V}EH \sin\theta$, or $\mathbf{R} = 0$.

10. Electric Radiation. Electric Waves. The damping of the mechanical vibrations described in § 45, III., C, XIII., was assumed to be due wholly to friction. A vibrating mechanical system, however, unless completely surrounded by a perfect vacuum, which is not possible, will set the adjacent parts of the surrounding medium into vibration, thus emitting a train of waves

Owing to the energy thus radiated to the surrounding medium, the motion of the system will be damped, and the damping so caused may greatly exceed the damping due to friction. This is true, for example, in the case of a vibrating air column, most of whose energy is emitted in waves of sound.

Other things being equal, it is clear that the damping due to radiation will be greater the greater the surface communicating the energy to the surrounding medium.

The damping of the electrical oscillations discussed in § 43 C, XIII., was also assumed to be due wholly to the dissipation of energy by resistance. But since the electromagnetic field of the system extends into all space, it is evident that when its oscillations, or variations in the nearer portions of its electromagnetic field, occur, a train of electromagnetic waves must be emitted by the system and propagated into space, and that the oscillations will therefore be damped owing to the energy thus radiated.

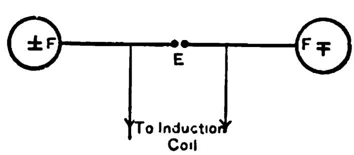


Fig. 141.

Other things being equal, it is clear that the damping due to radiation will be greater the farther into space the stronger parts of the system's electric and magnetic fields extend. Thus the system shown in Fig. 141, in which the fields spread far out into the surrounding dielectric, is a *good radiator*, the damping due to radiation being so great that only a few oscillations are completed; while a system such as that of Fig. 142, whose electric field is confined almost wholly to the region *AB*, and whose magnetic field is confined almost wholly to the region *CD*, is a *good vibrator*, but a *poor radiator*, its energy being completely radiated into space and dissipated in its resistance only after the execution of many oscillations.

At a considerable distance from an electrical oscillator the electromagnetic waves crossing a limited area will be approximately plane, just like the sound waves emitted by a vibrating bell or diapason.

In the electromagnetic wave train emitted by a symmetrical "dumb-bell" oscillator like that of Fig. 141, consisting of spheres at the ends of circular cylindrical rods, it is obvious that at any point the electric intensity will lie in the plane containing

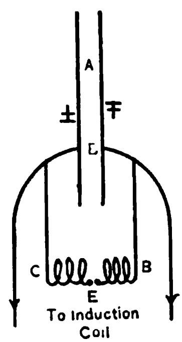


Fig. 142.

the point and passing through the axis of the oscillator, and the magnetic intensity will lie in the plane perpendicular to the axis, both intensities being perpendicular to the direction of propagation of the waves by Poynting's theorem.

Such a wave train passing through a given point at which the electric and magnetic intensities oscillate in fixed planes is said to be *plane polarised.*

Electromagnetic waves can also be developed by means of the convection of electric charges or magnetic poles. Thus if two equal spheres, with equal and opposite charges, are made to approach and recede from one another alternately, a wave system very similar to that of the "dumb-bell" oscillator will be emitted.

Also, if a system consisting of two equal spheres with equal and opposite charges, mounted upon an insulating rod, is rotated uniformly about an axis passing perpendicularly through the

center of the rod, another interesting and important wave system will be emitted. At any point along the axis of revolution each intensity, always perpendicular to this axis, remains constant in magnitude, but passes uniformly through all azimuths during every revolution. The radiation along the axis is therefore said to be *circularly polarised.* At all points in the plane of revolution of the centers of the spheres, the radiation is plane polarised. At points not in this plane and not on the axis of revolution, each intensity passes uniformly through all azimuths during each period, reaching a minimum and a maximum value twice each, but never becoming zero, and the radiation is therefore said to be *elliptically polarised.* The first and second cases are particular cases of the third.

On the electron theory, waves of light, which are electromagnetic waves of extremely short wave-length and period, are developed by the vibrations of the electrons, that is by electric convection, within the atom.

For the continuous production of electric waves, or rather for the rapid production of successive trains of such waves, the two conductors of the oscillator are connected permanently to the terminals of an electrical influence machine or induction coil in operation, as shown in the figures. Every time the voltage between the terminals of the oscillator reaches a certain value, the insulating properties of the dielectric break down along a line between the terminals, and the oscillations occur, the path of the current being evident from the spark. With the cessation of the oscillation the insulation is restored, the voltage again increases, a spark occurs, another wave train is emitted, and so on indefinitely.

For the detection of electric waves, any sufficiently sensitive electric vibrator may be employed. When used for this purpose, such an electrical system is called a *resonator.* One of the commonest forms of resonator is the dumb-bell form, similar to the dumb-bell vibrator, Fig. 141, but with a shorter spark gap E. If such a resonator is placed in a region traversed by electric waves with the rods EF parallel to the direction of the electric

intensity, or in any direction not perpendicular to the intensity, an e.m.f. of the same kind as that of the vibrator will be impressed upon the resonator parallel to its length, and oscillations will be set up therein. If the maximum value of the voltage developed between the terminals of the spark gap E is sufficient, the insulator within the gap will break down and the oscillations of the resonator will become manifest by the passage of a spark. This effect will be a maximum when the rods EF are parallel to the electric intensity of the waves, and zero when they are perpendicular to this intensity. For a given angle between the electric intensity and the axis of a resonator, the effect will also be a maximum when the period of the resonator is equal to that of the waves (or that of the vibrator), in accordance with the principles of § 44, XIII. One tuning fork set into resonant vibration by the waves from another in unison is an exact mechanical analogue.

The sensitiveness of a resonator can be greatly increased, or the minimum intensity for which it will give noticeable indications greatly diminished, by the addition of any one of several devices.

One of the most effective and widely used of these adjuncts is the *coherer* of Branly. This consists of a small glass tube plugged at the ends with metallic electrodes and loosely packed with metallic filings, or other small pieces of metal. The electrodes are connected to the resonator, usually on opposite sides of the spark gap, and also to the terminals of a circuit containing a battery and a galvanometer or other current indicator. Before the incidence of electric waves upon the resonator, the electric resistance of the coherer is very great and only a very small current traverses the galvanometer. But when oscillations are set up in the resonator by the impact of electric waves, the resistance of the coherer is greatly diminished, and the current through the galvanometer is therefore greatly increased. The resistance of the coherer retains its reduced value after the cessation of the waves, but the original high resistance can be immediately restored by tapping the instrument with a light hammer.

In practice this is done by a continuously operating hammer driven electrically by a separate circuit. The diminution of resistance on which the action of the coherer depends is probably brought about by the welding together of the metallic particles on the passing of very minute sparks between them when even very feeble oscillations are set up in the resonator. When the particles are shaken apart by the hammer, the resistance goes back to its previous magnitude.

An indication of the relative magnitude of the electric intensity is given by the maximum distance between the spark gap terminals of the resonator (which are made adjustable when the instrument is used for this purpose) at which sparking will occur; or, if a coherer is employed, by the diminution of resistance, or increase of galvanometer current, produced.

For a detailed account of the theory of electric waves and oscillations and of the extensive experimental investigations (in full accord with the theory) upon the subject, the reader is referred to Poincaré's *Les Oscillations Électriques*, J. J. Thomson's *Recent Researches in Electricity and Magnetism*, Winckelmann's *Handbuch*, and, for recent digests, to the *Rapports* of the International Congress of Physics, Vol. II. The most complete treatment of the theory of the propagation of waves along wires is contained in Heaviside's *Electromagnetic Theory* and *Electrical Papers*. For the electromagnetic theory of light, see Drude's *Optik*.

The following paragraphs contain the theory of some of the simplest and most fundamental phenomena of electric waves.

11. The Propagation of Electromagnetic Disturbances in a Non-Conducting Dielectric Containing no Other Electric or Magnetic Fields than Those of the Disturbance Itself. In this case the electric convection and conduction current densities are zero, and e and μ are constant in space and time. Hence the first and second laws of circuitation are

$$i = dD/dt = cdE/dt = \text{curl } H \qquad (11)$$

and

$$-dB/dt = \mu dH/dt = \text{curl } E \qquad (12)$$

(11) is equivalent to the three component equations

$$\left.\begin{aligned} c\,dE_1/dt &= dH_3/dy - dH_2/dz & (a)\\ c\,dE_2/dt &= dH_1/dz - dH_3/dx & (b)\\ c\,dE_3/dt &= dH_2/dx - dH_1/dy & (c) \end{aligned}\right\} \quad (13)$$

and (12) is equivalent to the three equations

$$\left.\begin{aligned} -\mu\,dH_1/dt &= dE_3/dy - dE_2/dz & (a)\\ -\mu\,dH_2/dt &= dE_1/dz - dE_3/dx & (b)\\ -\mu\,dH_3/dt &= dE_2/dx - dE_1/dy & (c) \end{aligned}\right\} \quad (14)$$

Simple Plane Wave. We shall consider first only the simplest of plane polarised electromagnetic waves, viz., a disturbance in which everywhere $E_2 = E_3 = 0$, and $E_1 (= E)$ is independent of x and y, or has the same magnitude and direction at all points of any plane distant z from XY plane, *i. e.*, a plane polarised plane wave. In this case (13) (a) becomes

$$dD_1/dt (= dD/dt) = c\,dE_1/dt (= c\,dE/dt) = -\,dH_2/dz \quad (15)$$

since the magnetic intensity must be independent of x and y when the electric intensity is independent of x and y; and (14) (b) becomes

$$dB_2/dt = \mu\,dH_2/dt = -\,dE_1/dz \quad (16)$$

Differentiating (15) with respect to t and (16) with respect to z, and combining the resulting equations, we obtain

$$d^2E_1/dt^2 = 1/c\mu \cdot d^2E_1/dz^2 \quad (17)$$

or

$$d^2D_1/dt^2 = 1/c\mu \cdot d^2D_1/dz^2 \quad (18)$$

Differentiating (15) with respect to z and (16) with respect to t and combining the resulting equations, we obtain

$$d^2H_2/dt^2 = 1/c\mu \cdot d^2H_2/dz^2 \quad (19)$$

or

$$d^2B_2/dt^2 = 1/c\mu \cdot d^2B_2/dz^2 \quad (20)$$

These four equations have all the same form and show that the electric and magnetic intensities and inductions are propa-

gated in a direction parallel to the axis of Z with the velocity

$$v = 1/(c\mu)^{\frac{1}{2}} \qquad (21)$$

To demonstrate this, we have only to obtain the general solution of one of the equations. Choosing (17), introducing two new variables

$$a = z - vt$$

and

$$b = z + vt$$

where v is given by (21), and eliminating z and t from (17) by means of these equations, we have

$$d^2E_1/dadb = 0, \text{ or } d/da\ (dE_1/db) = d/db\ (dE_1/da) = 0$$

Hence dE_1/da is a function of a only, and dE_1/db is a function of b only. Therefore E_1 consists of the sum of two terms, one a function of a only, and the other a function of b only. Thus the general solution of (17) is

$$E_1 = F_1(z - vt) + F_2(z + vt) \qquad (22)$$

where $F_1(z - vt)$ and $F_2(z + vt)$ are arbitrary functions of $(z - vt)$ and $(z + vt)$, respectively. Either function may be zero, but neither can be constant or contain a constant term, since a constant field is excluded by the conditions assumed above.

$F_1(z - vt)$ represents a disturbance in the dielectric traveling unchanged in form in the positive direction of Z with the velocity v. For at the time $t + t'$, F_1 has the same value

$$F_1[(z + vt') - v(t + t')] = F_1(z - vt)$$

at the plane whose Z coördinate is $z + vt'$ which, at the time t, it had at the plane whose Z coördinate is z.

Similarly, $F_2(z + vt)$ represents a disturbance traveling in the negative direction of Z with the same speed v.

At the time $t = 0$, (22) gives

$$E_0 = F_1(z) + F_2(z) \qquad (23)$$

If at the time $t = 0$, $dE_1/dt = 0$, or the initial electric field is static, (22) gives also

$$F_1'(z) = F_2'(z) \qquad (24)$$

Hence we have at the time $t = 0$ in this case, by integrating (24) and making use of (23),

$$F_1(z) = F_2(z) = \tfrac{1}{2}E_1 \tag{25}$$

there being no constant of integration, since there is no permanent field.

As an example, suppose that at the time $t = 0$, $E_1 = D_1/c = 2A \cos 2\pi z/L$ between the limits $z = +L/4$ and $z = -L/4$, and $E_1 = 0$ everywhere else; also that at the same time $dE_1/dt = 0$ everywhere. Then

$$F_1(z) = A \cos 2\pi z/L = F_2(z) = F(z)$$

and at any time t

$$\begin{aligned} E_1 = E = F(z - vt) + F(z + vt) \\ = A \cos 2\pi/L \cdot (z - vt) + A \cos 2\pi/L \cdot (z + vt) \end{aligned}$$

Thus at the time $t = 0$ the initial static displacement or intensity divides up into two equal waves, one running in the positive direction of Z with the speed v, and the other running in the

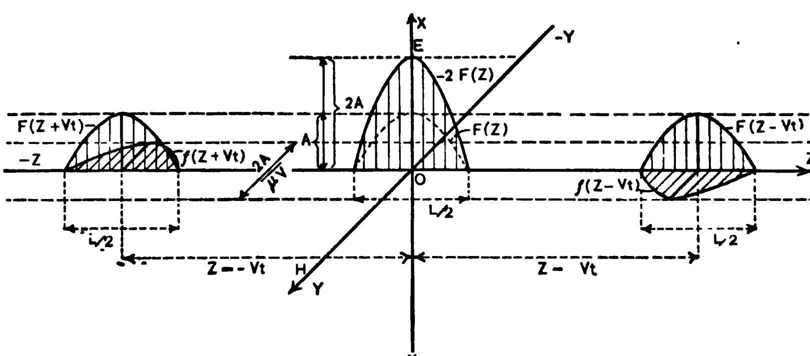


Fig. 143.

negative direction of Z with the same speed. At the time t the intensity is zero everywhere except between the planes $z = vt + L/4$ and $z = vt - L/4$ and between the planes $z = -(vt + L/4)$ and $z = -(vt - L/4)$. The initial disturbance and the disturbance at the time t are shown in Fig. 143.

12. The Relation Between E and H in the Electromagnetic Wave. From the exact similarity in the form of equations (17) and (19) it is now evident that

$$H_2 = f_1(z - vt) + f_2(z + vt) \tag{26}$$

where f_1 and f_2 are arbitrary functions of $(z - vt)$ and $(z + vt)$, respectively.

H_2 is the *total* magnetic intensity. For since $E_2 = E_3 = 0$, and $E_1 = E$ is independent of x and y, (14), (a) and (c) become

$$-\mu dH_1/dt = -\mu dH_3/dt = 0$$

which gives

$$H_1 = H_3 = 0$$

the constant of integration being zero, since there is no permanent field.

The arbitrary functions f_1 and f_2 are closely related to F_1 and F_2, as will appear from the following deduction of $H_2 = H$ from (16) and (22). From (16) we have

$$H_2(= H) = -1/\mu \int dE_1/dz\, dt \tag{27}$$

there being no constant of integration. Consider the disturbance

$$E_1 = F_1(z - vt) \tag{28}$$

traveling with the velocity v in the positive direction of Z. Differentiation of (28) gives

$$dE_1/dz = F_1'(z - vt)$$

which, substituted in (27), gives for the disturbance f_1 connected with F_1

$$\begin{aligned} H_2 = f_1(z - vt) &= -1/\mu \int F_1'(z - vt)dt \\ &= 1/\mu v \int dF_1(z - vt)/dt\, dt = 1/\mu v \cdot F_1(z - vt) \end{aligned} \tag{29}$$

In exactly the same way we obtain for the disturbance f_2 connected with F_2

$$H_2 = f_2(z+vt) = 1/\mu(-v)\cdot F_2(z+vt) = -1/\mu v\cdot F_2(z+vt) \quad (30)$$

which is exactly analogous to (29), since the velocity of the disturbance $f_2(z+vt)$ and $F_2(z+vt)$ is $-v$.

Thus in the general case, when $E = E_1$ is given by (22),

$$\begin{aligned} H = H_2 &= f_1(z-vt) + f_2(z+vt) \\ &= 1\mu v\cdot[F_1(z-vt) - F_2(z+vt)] \end{aligned} \quad (31)$$

$f_1(z-vt) + f_2(z+vt)$, for the times $t = 0$ and $t = t$, is shown in Fig. 143 for the case in which $F_1(z) = F_2(z) = A \cos 2\pi z/L$. Since $f_1(z) = -f_2(z) = cvF_1(z)$, the initial magnetic disturbance is zero.

With due regard to the directions as well as to the magnitudes of the electric and magnetic intensities and the velocity of the electromagnetic disturbance (F_1 and f_1, or F_2 and f_2), which will be denoted by v, both (29) and (30) give the relations

$$H_2(=H) = c\mathbf{V}vE_1 = c\mathbf{V}vE = \mathbf{V}vD \quad (32)$$

and

$$E_1(=E) = \mu\mathbf{V}H_2v = \mu\mathbf{V}Hv = \mathbf{V}Bv \quad (33)$$

since $v^2 = 1/c\mu$.

Quantitatively, either of these equations is equivalent to the relation

$$\tfrac{1}{2}\mu H^2 = \tfrac{1}{2}cE^2 \quad (34)$$

between the electric and magnetic energy densities.

The electromagnetic energy flux density at any point of the wave is

$$\mathsf{R} = \mathbf{V}EH = v(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2) \quad (35)$$

and has always the direction of the velocity of the wave.

13. A Plane Simple Harmonic Wave Train. As another example, we shall assume

$$E = E_1 = A \cos 2\pi/L\cdot(vt - z) \quad (36)$$

Then we have

$$H = H_2 = cvA \cos 2\pi/L\cdot(vt - z) \quad (37)$$

The relation between E, H, and the time, for the plane $z = 0$, as well as the relation between E, H, and z, for the time $t = 0$, is shown in the curves of Fig. 144.

The electromagnetic energy flux density at the time t across a plane distant z from the XY plane is

$$\mathbf{R} = \mathbf{V}EH = cvA^2 \cos^2 2\pi/L \cdot (vt - z) \tag{38}$$

E and H being given by (36) and (37).

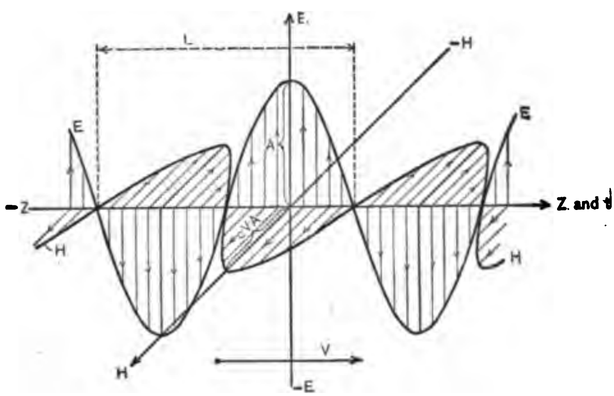


Fig. 144.

As shown by the equation, $\mathbf{R}$ is always positive except at points at which E and H, and therefore $\mathbf{V}EH = \mathbf{R}$, are zero. This is of course obvious, since the energy travels with the waves.

The mean value of $\mathbf{V}$ at any point during a complete period is

$$\begin{aligned}(\mathbf{R}) &= cvA^2 \times \text{mean value of } \cos^2 2\pi/L \cdot (vt - z) \\ &= cvA^2 \times \text{mean value of} \\ &\quad \tfrac{1}{2}[1 - \cos 4\pi/L \cdot (vt - z)] = cvA^2/2\end{aligned} \tag{39}$$

Since in a pure electromagnetic wave the electric and magnetic intensities travel with the same velocity v, the above result may also be obtained from the relation

$$\mathbf{R} = v(\tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2) = cvE^2 = \mu vH^2 \tag{40}$$

The mean value of E^2 is $A^2/2$, and the mean value of H^2 is $c^2v^2A^2/2$. Hence $(\mathbf{R}) = cvA^2/2$, as before.

14. A Plane Electromagnetic Wave Train in a Non-conducting Dielectric Incident and Reflected Normally at a Plane Interface Separating the Dielectric from (1) a Perfect Conductor or a Dielectric of Infinite Permittivity, or (2) a Perfect Insulator with Infinite Inductivity. (1) *Reflection from a perfect conductor.* Let the electric intensity in the incident wave train be denoted by

$$E_{1i} = A \cos (nt - pz) \tag{41}$$

the incident wave being propagated in the positive direction of Z.

If E_{1i} were the total intensity, the intensity at the interface would be $A \cos nt$, a quantity differing from zero except at two instants in every period. But since the conductivity of the conductor, or the permittivity of the second dielectric, is infinite, a finite intensity parallel to the interface would necessitate an infinite current in the conductor, according to Ohm's law, or an infinite displacement in the second dielectric, which is inadmissible. Hence there must be a reflected wave train whose intensity

$$E_{1r} = -A \cos (nt + pz) \tag{42}$$

added to the intensity E_{1i} of the incident train makes the total intensity

$$E_1 = E_{1i} + E_{1r} = A[\cos (nt - pz) - \cos (nt + pz)] \tag{43}$$

a quantity equal to zero, when $z = 0$, for all values of t.

The total magnetic intensity is

$$H_2 = H_{2i} + H_{2r} = p/\mu n \cdot A[\cos (nt - pz) + \cos (nt + pz)] \tag{44}$$

Equations (43) and (44) may be written

$$E_1 = 2A \sin nt \sin pz = 2A \sin 2\pi t/T \cdot \sin 2\pi z/L \tag{45}$$

and

$$H_2 = 2Ap/\mu n \cdot \cos 2\pi t/T \cdot \cos 2\pi z/L \tag{46}$$

Thus the incident and reflected wave trains *interfere* to produce a system of *standing waves*. The *electric nodes*, or points (planes)

at which the electric intensity and displacement are permanently zero, are distant from the interface o, $L/2$, L, $3L/2$, etc.; and the *antinodes*, or points (planes) at which the electric intensity reaches its maximum and minimum values, are distant from the interface $L/4$, $3L/4$, etc.

The magnetic nodes have the positions of the electric antinodes at distances $L/4$, $3L/4$, etc., from the interface; and the magnetic antinodes have the positions of the electric nodes at distances o, $L/2$, L, etc., from the interface. Thus the magnetic intensity is a maximum or minimum where the electric intensity is zero, and the electric intensity is a maximum or minimum where the magnetic intensity is zero.

(2) *Reflection from a perfect insulator with infinite inductivity.* In this case the magnetic intensity at the interface must be zero, since otherwise the induction ($B = \mu H$) in the dielectric with infinite inductivity, and the "magnetic current,"* would be infinite except at two instants in every period. Therefore the magnetic intensities of the incident and reflected wave trains must be equal and opposite at the interface. Hence, if we use the nomenclature of (1), we have

$$\begin{aligned} H_2 &= Ap/\mu n \cdot [\cos(nt - pz) - \cos(nt + pz)] \\ &= 2Ap/\mu n \cdot \sin 2\pi t/T \cdot \sin 2\pi z/L \end{aligned} \tag{47}$$

and

$$\begin{aligned} E_1 &= A\ [\cos(nt - pz) + \cos(nt + pz)] \\ &= 2A \cos 2\pi t/T \cdot \cos 2\pi z/L \end{aligned} \tag{48}$$

Thus the interference of the two trains of waves produces a system of standing waves in which the magnetic nodes and the electric antinodes are located at the interface and at distances $L/2$, L, etc., therefrom, while the magnetic antinodes and electric nodes are located at distances $L/4$, $3L/4$, $5L/4$, etc., from the interface. Thus the nodes in this case occupy the positions of the antinodes in (1), and the antinodes the positions of the nodes in (1).

* That is, the rate of change of magnetic flux, by analogy with dielectric current.

15. The Flux of Energy in a System of Standing Waves. For the electromagnetic energy flux density at the time t across a plane distant z from the XY plane, (45) and (46), or (47) and (48), § 14, give

$$\begin{aligned}\mathbf{R} = \mathbf{V}\, E_1 H_2 &= A^2 p/\mu n \cdot \sin 4\pi t/T \cdot \sin 4\pi z/L \\ &= cvA^2 \sin 4\pi t/T \cdot \sin 4\pi z/L\end{aligned} \tag{49}$$

since $n^2/p^2 = v^2 = 1/\mu c$.

Thus $\mathbf{R}$ is permanently zero at all points for which $\sin 4\pi z/L = 0$, that is at all electric and magnetic nodes (or antinodes). At any point between an electric node and a magnetic node, $\mathbf{R}$ goes through a complete cycle of positive and negative values in the time $T/2$, its amplitude being greatest at points for which $\sin 4\pi z/L = 1$, that is points half way between electric and magnetic nodes.

At any instant $\mathbf{R}$ has opposite signs on opposite sides of any node, and also on similar sides of successive nodes. During one quarter of a period the energy, wholly electrostatic at the start, streams from the electric antinodes (magnetic nodes) toward the electric nodes (magnetic antinodes), being completely transformed into magnetic energy at the end of the quarter period. During the next quarter period the energy, wholly magnetic at the beginning, streams from the magnetic antinodes toward the electric antinodes, being completely reconverted into electrostatic energy at the end of the quarter period. The energy density has now everywhere the same value as a half period earlier, but the sign of the electric intensity is everywhere opposite. During the next half period the same energy transfer and transformations occur, and at its close the electric intensity and the energy density have the same values as at the beginning of the period.

16. The Propagation of a Plane Simple Harmonic Wave Train in a Conducting Medium Containing no Other Electromagnetic Field than That of the Wave Train. In this case we have

$$i = kE + cdE/dt = \text{curl } H \tag{50}$$

and

$$-\mu dH/dt = \text{curl } E \tag{51}$$

from which the six component equations can be easily written down.

From these equations, by a process exactly analogous to that carried out in § 11, we obtain for the simple case in which $E_2 = E_3 = 0$, and $E_1 = E$ is independent of x and y,

$$k\mu dE_1/dt + \mu cd^2E_1/dt^2 = d^2E_1/dz^2 \tag{52}$$

and a similar equation for H_2, the other components of H being zero.

To solve (52) for the simple case of a harmonic wave train of given period T, or given wave-length L, progressing in the positive direction of Z with the velocity v (to be determined), assume

$$E_1 = E = Ae^{-mz} \cos 2\pi/L\cdot(vt - z) \tag{53}$$

For the sake of brevity put

$$n = pv = 2\pi/T = 2\pi v/L \tag{54}$$

then (53) becomes

$$E_1 = E = Ae^{-mz} \cos (nt - pz) \tag{55}$$

The damping factor e^{-mz} is inserted on account of the dissipating effect of resistance, m being a quantity to be determined.

Substituting for E_1 in (52) its value as given by (55), and equating to zero separately the coefficients of sin $(nt - pz)$ and cos $(nt - pz)$ in the resulting equation, we obtain as the conditions that (55) may be a solution of (52),

$$-n^2\mu c - m^2 + p^2 = 0$$

and

$$-\mu nk + 2mp = 0$$

Hence

$$m = n\{-\mu c/2 + [(\mu c/2)^2 + (\mu k/2n)^2]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{56}$$

and

$$p = n\{\mu c/2 + [(\mu c/2)^2 + (\mu k/2n)^2]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{57}$$

both m and p being positive quantities, since the waves are damped and since they progress in the positive direction of z.

From (54) and (57), the velocity of the waves is

$$v = n / p = 1 / \{\mu c / 2 + [(\mu c / 2)^2 + (\mu k / 2n)^2]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{58}$$

being thus a function of n as well as of μ, c, and k.

The total magnetic intensity is

$$\begin{aligned} H = H_2 &= -1/\mu \int^t dE_1 / dz\, dt \\ &= A / \mu n \cdot e^{-mz}[p \cos(nt - pz) + m \sin(nt - pz)] \end{aligned} \tag{59}$$

If we put

$$p = N \cos \theta \text{ and } m = N \sin \theta$$

we have

$$N = (m^2 + p^2)^{\frac{1}{2}}$$

and

$$\theta = \tan^{-1} m / p$$

By means of these equations (59) may be written

$$H_2 = A(m^2 + p^2)^{\frac{1}{2}} / \mu n \cdot \cos(nt - pz - \theta) \tag{60}$$

For good conductors, such as liquid electrolytes and metals, the ratio $(\mu c/2)/(\mu k/2n) = cn/k$ is exceedingly small except for enormous values of n. Thus, even for so great a value of n as 10^6, cn/k for common aqueous solutions of salts and acids is of the order 10^{-3}, and for metallic conductors is too small to have been detected by experiment. For good conductors, therefore, we may write without sensible error, except for enormous values of n,

$$m = p = n(\mu k/2n)^{\frac{1}{2}} = (\mu kn/2)^{\frac{1}{2}}$$

and

$$\theta = \tan^{-1} m/p = \tan^{-1} 1 = \pi/4 \tag{61}$$

(55) and (59) thus become

$$E_1 = Ae^{-(\mu kn/2)^{\frac{1}{2}} z} \cos n[t - (\mu k/2n)^{\frac{1}{2}} z] \tag{62}$$

and

$$H_2 = A(k/\mu n)^{\frac{1}{2}} e^{-(\mu kn/2)^{\frac{1}{2}} z} \cos \{n[t - (\mu k/2n)^{\frac{1}{2}} z] - \pi/4\} \tag{63}$$

The velocity of the wave train in the conductor is

$$v = n/p = (2n/\mu k)^{\frac{1}{2}} \tag{64}$$

The relations between $E_1 = E$, $H_2 = H$, and z at the time $t = 0$, are similar to the relations between q, I, and t, Fig. 119.

At a distance z from the origin the amplitudes of the electric and magnetic intensities are less than their amplitudes at the origin in the ratio $e^{-(\mu k n/2)^{\frac{1}{2}} z}$ to 1. The distance $1/m = (2/\mu k n)^{\frac{1}{2}}$, in which the amplitude of either wave is reduced to $1/e$ of its value at the origin, is called the *relaxation distance* for the given medium and the given value of n. The distance in which the amplitude of either intensity falls to any fraction of its value at the origin is, like the relaxation distance $1/m$, inversely proportional to μ, k, and n. Thus if μ, k, or n is very great the intensity of the waves falls off very rapidly. If either k, μ, or n is infinite, that is if the conductor is a perfect conductor, its inductivity infinite, or the frequency of the waves infinite, all ideal cases, the electromagnetic disturbance does not enter the conductor at all. Thus a perfect conductor or a medium of infinite inductivity would form a perfect electric and magnetic screen in either a static or a variable electric or magnetic field.

For copper, when $n = 2\pi \times 100$, $m = \pi/2$ approximately. In this case the amplitudes of the intensities are reduced to the fractions 0.208, 0.043, and less than 1/500 the origin values at the distances 1, 2, and 4 cms., respectively, from the origin. When $n = 2\pi \times 1{,}000{,}000$, $m = 50\pi$ approximately, and the amplitudes are reduced to less than 1/6,000,000 the origin values in going a distance of 1 mm.

In the case of iron, if $\mu = 1000$, $m = 20$, approximately, when $n = 2\pi \times 100$. The amplitude of either intensity falls off to about thirteen hundredths and one twenty-thousandth part of the value at the origin in traversing the distances 1 mm. and 5 mm., respectively. When $n = 2\pi \times 1{,}000{,}000$, $m = 2000$ approximately, and the amplitudes fall off in 1/10 mm. to about one five hundred millionth part of their values at the origin.

These examples are given by J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, p. 418.

17. The Propagation of an Electromagnetic Field into a Conducting Cylinder. The magnetic field of a long circular solenoid traversed by a steady current is described in § 20, XII. The electric field, if the resistance of the solenoid is small, is weak, and the only flux of energy into the solenoid is the flux developing the Joulean heat in the wire. In what follows the resistance of the solenoid will be supposed very small and its counter e.m.f. negligible in comparison with the e.m.f. of induction. The radius of the solenoid will be denoted by a.

If the magnetic flux through the coil varies, that is if tubes of magnetic induction move outwards or inwards, an electric field will be developed within and without the solenoid. The lines of electric intensity will be circles centered on the axis in planes perpendicular thereto, and the e.m.f. around any circle of radius r will be given, in magnitude and direction, by

$$2\pi r E = - d\Phi / dt$$

where $d\Phi/dt$ is the rate at which magnetic flux in the positive direction crosses the circle inwardly, or by

$$2\pi r E = 2\pi r \mathbf{V} B u$$

where u is the velocity of the tubes of magnetic induction at the circle of radius r. The electric intensity is always zero on the axis.

The energy flux density, whose direction coincides with the direction of motion of the electric and magnetic tubes, is $R = \mathbf{V} E H$, which is always radial, toward or from the axis.

If now an alternating e.m.f. acts upon the coil, the electric and magnetic inductions will be propagated inwards and outwards alternately, the direction, as well as the direction of motion, of each being reversed once every half period.

If the period of the alternation is large, so that the distance traversed by the tubes of induction during one period is great in

comparison with the radius of the solenoid and the radii of all circles of electric intensity considered (all supposed small in comparison with the length of the solenoid), the magnetic induction will have sensibly the same value throughout the solenoid at any instant. The electric intensity at a distance r from the axis will be approximately

$$E = -\pi r^2/\pi a^2 \cdot (d\Phi/dt)/2\pi r = r(d\Phi/dt)/2\pi a^2$$

if r is less than a; and

$$E = -(d\Phi/dt)/2\pi r$$

if r is greater than a; Φ being total flux in the positive direction through the solenoid at the time t.

If the period of the alternation is small, so that the distance traversed by the tubes during one period is of similar magnitude to that of the radius of the solenoid, then the magnitude and direction of both intensities will vary with r.

If in addition the core of the solenoid is a conductor, or if it exhibits hysteresis, or both, as when made of iron, then the energy of the tubes will be partly dissipated during their propagation in the core. Hence, since the direction of each intensity is periodically reversed, the amplitude of the magnetic intensity as well as that of the electric intensity, will steadily diminish as the axis is approached.

If the radius of the core is great, or the curvature of its surface small, the law of the diminution of the intensities with the distance from the surface is approximately the same as that deduced for a conductor traversed by plane waves, § 16, the ratio of the amplitude of either intensity at the distance z from the surface to its surface value being approximately $e^{-(\mu kn/2)^{\frac{1}{2}}z}/1$.

Precisely the same form of reasoning applies to the propagation of an alternating electromagnetic field into the cylindrical conductor of § 3.

The same form of reasoning also applies to a cylindrical conductor (§ 6) to whose surface the electric intensity is not parallel,

since the parallel component only is concerned in the propagation of energy into the conductor.

Thus a rapidly alternating current is not distributed uniformly throughout the conductor, but is more or less concentrated near its surface. This increases the resistance of the conductor, and decreases its inductance, the former being greater the less the area of the section across which the current flows, and the latter being less the thinner and farther from the axis the walls of the tube through which the principal part of the current now flows [§ 21, (2), XIII.].

In the case of a very thin wire, all points of the surface are very near to the axis, hence both the above effects are small.

18. The Propagation of Waves Along Wires. Since the electromagnetic waves discussed in §§ 11–15 are propagated unchanged at right angles to the intensities, it is clear that the results there obtained hold good for any plane plane polarised wave, whether the wave front is infinite or not and whether the direction of E (as well as that of H) is the same for all parts of the wave front or not.

Thus they apply to plane waves propagated between two parallel perfectly conducting planes, or to waves propagated along two parallel perfectly conducting cylinders concentric like those of §§ 22, XIII., and 1, or side by side like those of § 24, XIII. That the electric and magnetic intensities of these systems are perpendicular at any point in the case of electric waves as in the case of a steady current is apparent from previous discussions without reference to (32) or (33), § 12.

If the resistance of the conductors is not zero, but small, the relations deduced in the articles referred to above will apply approximately. Thus electric waves travel along wires of small resistance surrounded by a given medium with approximately the same velocity as in free space containing the same medium.

If the two wires are joined at the end remote from the oscillator by a large plane conductor perpendicular to their lengths,

or even by bending them together, a system of standing waves, resembling that of § 14, (1) will result, with an approximate electric node and magnetic antinode at this end.

If, on the other hand, the wires are insulated from one another at this end, a system of standing waves, resembling that of § 14, (2) will result, with an approximate magnetic node and electric antinode at this end.

19. Mechanical Analogue of an Electromagnetic Wave. Waves in Frictionless Elastic Media. Consider a plane transverse wave traversing an infinite elastic medium in the positive direction of Z. Let the displacement of the medium take place parallel to the Y

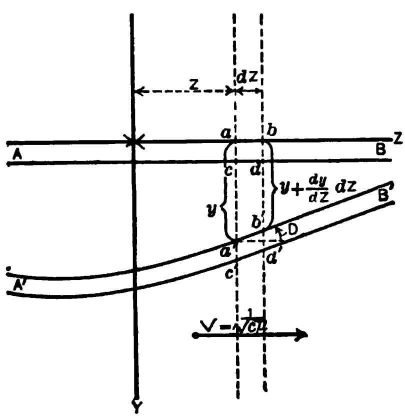


Fig. 145.

axis. Then at the time t while the disturbance is crossing any plane distant z from the XY plane, every point of the medium in this plane will be shifted in the same direction and through the same distance, y, from its equilibrium position.

Fig. 145 shows a section parallel to the YZ plane of a portion of the medium in its undisturbed state AB, and in a disturbed state $A'B'$ at the time t while a wave is passing. Every infinitesimal parallelepiped with its sides parallel to the coördinate planes which is bounded on the sides parallel to XY by planes distant z and $z + dz$ from the XY plane is shifted and sheared

precisely like the parallelepiped *abcd*, which is shifted and sheared into the parallelepiped $a'b'c'd'$.

Let the shear (equal to the angle between *ab* and $a'b'$) at the plane z be denoted by D. Then

$$D = - dy/dz \tag{65}$$

the negative sign being chosen since D, which is a vector perpendicular to *ab* and $a'b'$, is positive, as in the case represented in the figure, when a right-handed screw rotating from *ab* to $a'b'$ would move in the positive direction of X (up from the plane of the paper in the figure).

Let the area of each face of the parallelepiped parallel to the XY plane be denoted by dS; and let the modulus of rigidity, or shear modulus, of the medium be denoted by $n = 1/c$, c being the shear permittivity. Then, if E denotes the shearing stress in the plane z,

$$E = nD = - ndy/dz = - 1/c \cdot dy/dz \tag{66}$$

The shearing force upon the face $a'c'$ is therefore

$$EdS = nDdS$$

and that upon the face $b'd'$ is

$$-(E + dE/dz\, dz)dS = -n(D + dD/dz\, dz)dS$$

the force being positive when directed in the positive direction of the Y axis. The total force upon the parallelepiped $a'b'c'd'$ is therefore

$$- dE/dz\, dzdS$$

Let the density of the medium be denoted by μ. Then, since the volume of the parallelepiped is $dz\, dS$, its mass is

$$\mu dzdS$$

Let the velocity of the parallelepiped be denoted by H (positive when in the positive direction of Y). Then

$$H = dy/dt \tag{67}$$

Hence, by the second law of motion,

$$\mu dz\, dS\, dH/dt = -dE/dz\, dz\, dS$$

whence

$$\mu dH/dt = -dE/dz \tag{68}$$

By differentiating (66) with respect to t we obtain

$$\begin{aligned} cdE/dt = dD/dt &= -d/dt\cdot(dy/dz) \\ &= -d/dz\cdot(dy/dt) = -dH/dz \end{aligned} \tag{69}$$

By differentiating (69) with respect to t and (68) with respect to z and combining the resulting equations we obtain

$$c\mu d^2E/dt^2 = d^2E/dz^2 \tag{69}$$

or

$$c\mu d^2D/dt^2 = d^2D/dt^2 \tag{70}$$

By differentiating (69) with respect to z and (68) with respect to t and combining the resulting equations we obtain

$$c\mu d^2H/dt^2 = d^2H/dz^2 \tag{71}$$

or

$$c\mu d^2B/dt^2 = d^2B/dz^2 \tag{72}$$

Equations (69)–(72) are identical with equations (17)–(20), and show that all the results of §§ 11–15 apply also to the dynamical waves here considered, the shear permittivity and density of an elastic medium being substituted for the electric permittivity and magnetic inductivity of a dielectric, shearing stress and shear for electric intensity and displacement, and linear velocity and momentum per unit volume for magnetic intensity and induction.

By introducing internal friction, the analogy may be readily extended to the damped electromagnetic waves of § 16.

20. The Stresses in an Electromagnetic Wave. Electromagnetic Radiation Pressure. At every point in an electromagnetic wave there is a pressure normal to the plane containing the electric and magnetic intensities, that is normal to the wave front,

equal to the sum of the electric and magnetic pressures, or the sum of the electric and magnetic energy densities, at the point, by §§ 40–41, I., and § 18, XI. Thus, if p denotes this pressure,

$$p = \tfrac{1}{2}cE^2 + \tfrac{1}{2}\mu H^2 \tag{73}$$

In a *single* wave, or train of waves, $\frac{1}{2}cE^2 = \frac{1}{2}\mu H^2$ at any point, and, as seen in all the electromagnetic waves considered above, the electric and magnetic intensities at any point are perpendicular to one another. Hence the electric tension in the direction of the electric intensity is just neutralised by the magnetic pressure perpendicular to the magnetic intensity, and the magnetic tension in the direction of the magnetic intensity is just neutralised by the electric pressure perpendicular to the electric intensity. Thus the pressure p normal to the plane of the intensities is the total (dynamical) stress in the wave.

In wave systems in which at any point $\frac{1}{2}cE^2$ is not equal to $\frac{1}{2}\mu H^2$, as the systems of § 14, there is in addition to the normal pressure p, a tension $\frac{1}{2}cE^2 - \frac{1}{2}\mu H^2$, or a pressure $\frac{1}{2}\mu H^2 - \frac{1}{2}cE^2$, parallel to the electric intensity, and a tension $\frac{1}{2}\mu H^2 - \frac{1}{2}cE^2$, or a pressure $\frac{1}{2}cE^2 - \frac{1}{2}\mu H^2$, parallel to the magnetic intensity.

If electromagnetic waves in a given dielectric (1) are incident normally upon the interface separating this dielectric from another medium (2), at the surface of which, or within which, the intensities, and therefore the pressures, are reduced to zero (by total reflection from the interface, partial reflection and partial absorption, or total absorption), the interface will, in accordance with what precedes, experience a force directed toward medium (2) and equal to $p \times$ the area of the interface exposed to the waves. If the waves are partially transmitted *through* medium (2), emerging at a second interface, the total pressure upon medium (2) in the direction of the propagation of the incident waves is equal to the difference between the values of p at the two interfaces.

If medium (2) is a perfect conductor, the waves are totally reflected, the electric intensity at the interface is zero, and the magnetic intensity at the interface is twice the magnetic intensity of

the incident wave, that is, $2\mathbf{H}\cos nt$, if $\mathbf{H}\cos nt$ denotes the magnetic intensity at the interface of the incident wave. Thus the radiation pressure upon the interface at the time t is

$$p = \tfrac{1}{2}\mu(2\mathbf{H}\cos nt)^2 = 2\mu\mathbf{H}^2\cos^2 nt \tag{74}$$

and the mean value of the pressure during a complete period is

$$\begin{aligned}(p) &= 2\mu\mathbf{H}^2 \times \text{mean value of } \cos^2 nt \\ &= 2\mu\mathbf{H}^2 \times \text{mean value of } (\tfrac{1}{2} + \tfrac{1}{2}\cos 2nt) = \mu\mathbf{H}^2\end{aligned} \tag{75}$$

If medium (2) is a non-conductor with infinite inductivity, the waves are totally reflected, the magnetic intensity at the interface is zero permanently, and the electric intensity there is twice the electric intensity of the incident wave, that is $2\mathbf{E}\cos nt$, if $\mathbf{E}\cos nt$ denotes the electric intensity of the incident wave at the interface. Thus the pressure upon the interface at the time t is

$$p' = \tfrac{1}{2}c(2\mathbf{E}\cos nt)^2 = 2c\mathbf{E}^2\cos^2 nt = p \tag{76}$$

and the mean value of p' during a complete period is

$$(p') = c\mathbf{E}^2 = (p) \tag{77}$$

If the energy of the incident wave is totally absorbed by medium (2), there is no reflected wave, and the pressure upon medium (2) is

$$\begin{aligned}p'' &= \tfrac{1}{2}c(\mathbf{E}\cos nt)^2 + \tfrac{1}{2}\mu(\mathbf{H}\cos nt)^2 \\ &= \tfrac{1}{2}(c\mathbf{E}^2 + \mu\mathbf{H}^2)\cos^2 nt = \tfrac{1}{2}p' = \tfrac{1}{2}p\end{aligned} \tag{78}$$

The mean value of the pressure during a complete period is

$$(p'') = \tfrac{1}{4}(c\mathbf{E}^2 + \mu\mathbf{H}^2) = \tfrac{1}{2}(p') = \tfrac{1}{2}(p) \tag{79}$$

For experimental investigations confirming in a striking manner the theory of radiation pressure, developed independently and in different ways by Maxwell and Bartoli, see P. Lebedew, *Ann. der Physik*, Vol. 6, p. 433, 1901; and especially Nichols and Hull, *Astrophysical Journal*, Vol. 17, p. 315, 1903.

For the theory of vibration pressure in general, see Lord Rayleigh, *Philosophical Magazine*, Vol. 3, p. 338, 1902.

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